

STOCHASTIC AND PARTIAL DIFFERENTIAL EQUATIONS ON NON-SMOOTH TIME-DEPENDENT DOMAINS

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ABSTRACT. In this article, we consider the setting of single-valued, smoothly varying directions of reflection and non-smooth time-dependent domains whose boundary is Hölder continuous in time. In this setting, we prove existence and uniqueness of strong solutions to stochastic differential equations with oblique reflection. In the same setting, we also prove, using the theory of viscosity solutions, a comparison principle for viscosity solutions to fully nonlinear second-order parabolic partial differential equations and, as a consequence, we obtain existence and uniqueness for this class of equations as well. Our results are generalizations of two articles by Dupuis and Ishii to the setting of time-dependent domains.

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1. INTRODUCTION

In this article we establish existence and uniqueness of solutions to stochastic differential equations (SDE) with oblique reflection at the boundary of a bounded, time-dependent domain. In the same geometric setting, we also prove a comparison principle, implying uniqueness and existence results, of solutions to partial differential equations (PDE) with oblique derivative boundary conditions.

In the SDE case, our approach is based on the Skorohod problem, which, in the form studied in this article, was first described by Tanaka [21] who established existence and uniqueness of solutions to the Skorohod problem in convex domains with normal reflection. These results were subsequently substantially generalized by, in particular, Lions and Sznitman [17] and Saisho [20]. The most general results on strong solutions to reflected SDEs based on the Skorohod problem are those established by Dupuis and Ishii [11]. In the PDE case, our approach is based on the nowadays standard theory of viscosity solutions. A similar approach was used by Dupuis and Ishii to prove existence and uniqueness of viscosity solutions to oblique derivative problems for fully nonlinear second-order elliptic PDEs in non-smooth bounded domains [9]. Dupuis and Ishii also studied oblique derivative problems for fully nonlinear elliptic PDE's on domains with corners in [10].

The aim of this article is to generalize the SDE and PDE results mentioned above to the setting of time-dependent domains and, in the PDE setting, to parabolic PDEs. The topic of reflected SDEs on time-dependent domains was first studied by Costantini, Gobet and El Karoui [6] on smooth domains with reflection in the normal direction. Existence of weak solutions for non-smooth time-dependent domains with oblique reflection was established by Nyström and Önskog [19] under fairly general conditions. In this article we use the approach of [11] and derive regularity conditions, under which we can obtain existence and uniqueness of solutions to SDEs with oblique reflection on time-dependent domains. To connect

to the PDE case, we recall that the approach of [11] relies on the construction of test functions used earlier in [9] to prove the comparison principle, existence and uniqueness, in the PDE case. Here we generalize these test functions to our setting, and obtain more general results for both SDEs and PDEs.

To the authors' knowledge, there are no previous results on the oblique derivative problem for parabolic PDEs in non-smooth time-dependent domains. In time-independent domains however, there are several papers in the literature. For example, Barles [2] proved a comparison principle and existence of unique solutions to degenerate elliptic and parabolic boundary value problems with nonlinear Neumann type boundary conditions in bounded domains with $W^{3,\infty}$ -boundary. Ishii and Sato [15] proved similar theorems for boundary value problems for some singular degenerate parabolic partial differential equations with nonlinear oblique derivative boundary conditions in bounded C^1 -domains. Moreover, in bounded domains with $W^{3,\infty}$ -boundary, Bourgoing [4] considered singular degenerate parabolic equations and equations having L^1 dependence in time.

Concerning PDEs in the setting of time-dependent domains, we mention that Björn *et al.* [3] proved, among other results, a comparison principle for solutions of degenerate and singular parabolic equations using a different technique. Moreover, Avelin [1] proved boundary estimates of solutions to the degenerate p -parabolic equation, while Burdzy, Zhen and Sylvester [5] considered the heat equation and reflected Brownian motion in time-dependent domains.

As a motivation for considering SDEs and PDEs in time-dependent domains, we mention that such geometries arise naturally in a wide range of applications in which the governing equation of interest is a differential equation. For example, modeling of crack propagation [18], modeling of fluids [12], [13] and modeling of chemical, petrochemical and pharmaceutical processes [16].

The rest of the paper is organized as follows. In Section 2 we give preliminary definitions, notations, assumptions and also state our main results. In Section 3 we construct test functions crucial for the proofs of both the SDE and the PDE results. Using these test functions, we prove existence of solutions to the Skorohod problem in Section 4 and the results on the Skorohod problem are subsequently used, in Section 5, to prove the main results for SDEs. Finally, in Section 6, we use the theory of viscosity solutions together with the test functions derived in Section 3 to establish the main PDE results.

2. PRELIMINARIES AND STATEMENT OF MAIN RESULTS

Throughout this article we will use the following definitions and assumptions. Given $n \geq 1$, $T > 0$ and a bounded, open, connected set $\Omega' \subset \mathbb{R}^{n+1}$ we will refer to

$$(2.1) \quad \Omega = \Omega' \cap ([0, T] \times \mathbb{R}^n),$$

as a time-dependent domain. Given Ω and $t \in [0, T]$, we define the time sections of Ω as $\Omega_t = \{x : (t, x) \in \Omega\}$, and we assume that

$$(2.2) \quad \Omega_t \neq \emptyset \text{ and that } \Omega_t \text{ is bounded and connected for every } t \in [0, T].$$

Let $\partial\Omega_t$, for $t \in [0, T]$, denote the boundary of Ω_t . Let $\langle \cdot, \cdot \rangle$ and $|\cdot| = \langle \cdot, \cdot \rangle^{1/2}$ define the Euclidean inner product and norm, respectively, on \mathbb{R}^n and define, whenever $a \in \mathbb{R}^n$ and $b > 0$, the sets $B(a, b) = \{x \in \mathbb{R}^n : |x - a| \leq b\}$ and $S(a, b) = \{x \in \mathbb{R}^n : |x - a| = b\}$. For any Euclidean spaces E and F , we define the following spaces of functions mapping E into F . $\mathcal{C}(E, F)$ denotes the set of continuous functions, $\mathcal{C}^k(E, F)$ denotes the set of k times continuously differentiable functions and $\mathcal{H}^1(E, F)$ denotes a Sobolev space of one time weakly differentiable functions. If we can distinguish the time variables from the spatial variables, we

let $\mathcal{C}^{1,2}(E, F)$ denote the set of functions, whose elements are continuously differentiable once with respect to the time variable and twice with respect to any space variable. Moreover, $\mathcal{BV}(E, F)$ denotes the set of functions with bounded variation. In particular, for $\eta \in \mathcal{BV}([0, T], \mathbb{R}^n)$, we let $|\eta|(t)$ denote the total variation of η over the interval $[0, t]$.

2.1. Assumptions on the domain and directions of reflection. Throughout this article we consider non-smooth time-dependent domains of the following type. Let $\Omega \subset \mathbb{R}^{n+1}$ be a time-dependent domain satisfying (2.2). The direction of reflection at $x \in \partial\Omega_t$, $t \in [0, T]$ is given by $\gamma(t, x)$, where

$$(2.3) \quad \gamma \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^n, S(0, 1)).$$

Moreover, there is a constant $\rho \in (0, 1)$, such that the exterior cone condition

$$(2.4) \quad \bigcup_{0 \leq \zeta \leq \rho} B(x - \zeta \gamma(t, x), \zeta \rho) \subset \Omega_t^c,$$

holds, for all $x \in \partial\Omega_t$, $t \in [0, T]$. Note that it follows from (2.4) that γ points into the domain and this is indeed the standard convention for SDEs. For PDEs, however, the standard convention is to let γ point out of the domain. To facilitate for readers accustomed with either of these conventions we, in the following, let γ point inward whenever SDEs is treated, whereas when we treat PDEs we assume the existence of a function

$$(2.5) \quad \tilde{\gamma} \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^n, S(0, 1)),$$

such that

$$(2.6) \quad \bigcup_{0 \leq \zeta \leq \rho} B(x + \zeta \tilde{\gamma}(t, x), \zeta \rho) \subset \Omega_t^c,$$

for all $x \in \partial\Omega_t$, $t \in [0, T]$. Finally, regarding the temporal variation of the domain, we define $d(t, x) = d(x, \Omega_t)$, for all $t \in [0, T]$, $x \in \mathbb{R}^n$, and assume that

$$(2.7) \quad d(\cdot, x) \in \mathcal{H}^1([0, T], [0, \infty)).$$

for all $x \in \mathbb{R}^n$.

Remark 2.1. *A simple contradiction argument based on the exterior cone condition (2.4) for the time sections and the regularity of γ and Ω_t , shows that the time sections satisfy the interior cone condition*

$$\bigcup_{0 \leq \zeta \leq \rho} B(x + \zeta \gamma(t, x), \zeta \rho) \subset \overline{\Omega}_t,$$

for all $x \in \partial\Omega_t$, $t \in [0, T]$. The exterior and interior cone conditions together imply that the boundary of Ω_t is Lipschitz continuous (in space) with a Lipschitz constant K_t satisfying $\sup_{t \in [0, T]} K_t < \infty$. Moreover, these conditions imply that for a suitable constant $\theta \in (0, 1)$, $\theta^2 > 1 - \min_{t \in [0, T]} (\rho(t))^2$, there exists $\delta > 0$ such that

$$\langle y - x, \gamma(t, x) \rangle \geq -\theta |y - x|,$$

for all $x \in \partial\Omega_t$, $y \in \overline{\Omega}_t$, $t \in [0, T]$ satisfying $|x - y| \leq \delta$.

Remark 2.2. *By the Sobolev embedding theorem, condition (2.7) implies the existence of a positive constant $K < \infty$ such that*

$$(2.8) \quad |d(s, x) - d(t, x)| \leq K |s - t|^{1/2},$$

for $s, t \in [0, T]$, $x \in \mathbb{R}^n$.

Remark 2.3. Consider the function

$$l(r) = \sup_{\substack{s,t \in [0,T] \\ |s-t| \leq r}} \sup_{x \in \overline{\Omega}_s} d(x, \Omega_t),$$

introduced in [6] and frequently used in [19]. Condition (2.8) is equivalent to,

$$l(r) \leq K\sqrt{r},$$

which is considerably stronger than the condition $\lim_{r \rightarrow 0^+} l(r) = 0$, which was assumed in [19]. On the other hand, it was assumed in [19] that Ω_t satisfies a uniform exterior sphere condition, and this does not hold in general for domains satisfying (2.4).

2.2. Statement of main result for SDEs. We consider the Skorohod problem in the following form.

Definition 2.4. Given $\psi \in \mathcal{C}([0, T], \mathbb{R}^n)$, with $\psi(0) \in \overline{\Omega}_0$, we say that the pair $(\phi, \lambda) \in \mathcal{C}([0, T], \mathbb{R}^n) \times \mathcal{C}([0, T], \mathbb{R}^n)$ is a solution to the Skorohod problem for (Ω, γ, ψ) if (ψ, ϕ, λ) satisfies, for all $t \in [0, T]$,

$$(2.9) \quad \phi(t) = \psi(t) + \lambda(t), \quad \phi(0) = \psi(0),$$

$$(2.10) \quad \phi(t) \in \overline{\Omega}_t,$$

$$(2.11) \quad |\lambda|(T) < \infty,$$

$$(2.12) \quad |\lambda|(t) = \int_{(0,t]} I_{\{\phi(s) \in \partial\Omega_s\}} d|\lambda|(s),$$

$$(2.13) \quad \lambda(t) = \int_{(0,t]} \widehat{\gamma}(s) d|\lambda|(s),$$

for some measurable function $\widehat{\gamma} : [0, T] \rightarrow \mathbb{R}^n$ satisfying $\widehat{\gamma}(s) = \gamma(s, \phi(s)) d|\lambda|$ -a.s.

We use the Skorohod problem to construct solutions to SDEs confined to the given time-dependent domain $\overline{\Omega}$ and with direction of reflection given by γ . We shall consider the following to notion of SDEs. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration satisfying the usual conditions. Let m be a positive integer, let $W = (W_i)$ be an m -dimensional Wiener process and let $b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ be continuous functions.

Definition 2.5. A strong solution to the SDE in $\overline{\Omega}$ driven by the Wiener process W and with coefficients b and σ , direction of reflection along γ and initial condition $x \in \overline{\Omega}_0$ is an $\{\mathcal{F}_t\}$ -adapted continuous stochastic process $X(t)$ which satisfies, \mathbb{P} -almost surely, whenever $t \in [0, T]$,

$$(2.14) \quad X(t) = x + \int_0^t b(s, X(s)) ds + \int_0^t \langle \sigma(s, X(s)), dW(s) \rangle + \Lambda(t),$$

where

$$(2.15) \quad X(t) \in \overline{\Omega}_t, \quad |\Lambda|(t) = \int_{(0,t]} I_{\{X(s) \in \partial\Omega_s\}} d|\Lambda|(s) < \infty,$$

and where

$$(2.16) \quad \Lambda(t) = \int_{(0,t]} \widehat{\gamma}(s) d|\Lambda|(s),$$

for some measurable function $\widehat{\gamma} : [0, T] \rightarrow \mathbb{R}^n$ satisfying $\widehat{\gamma}(s) = \gamma(s, X(s)) d|\Lambda|$ -a.s.

Comparing Definitions 2.4-2.5, it is clear that $(X(\cdot), \Lambda(\cdot))$ should solve the Skorohod problem for $\psi(\cdot) = x + \int_0^\cdot b(s, X(s)) ds + \int_0^\cdot \langle \sigma(s, X(s)), dW(s) \rangle$ on an a.s.

pathwise basis. We assume that the coefficient functions $b(t, x)$ and $\sigma(t, x)$ satisfy the Lipschitz continuity condition

$$(2.17) \quad |b(t, x) - b(t, y)| \leq K|x - y| \quad \text{and} \quad \|\sigma(t, x) - \sigma(t, y)\| \leq K|x - y|,$$

for some positive constant $K < \infty$. Here $\|\cdot\|$ represents the Frobenius norm. Our main result is the following theorem.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a time-dependent domain satisfying (2.1) and assume that (2.3)-(2.4), (2.7) and (2.17) hold. Then there exists a strong solution to the SDE in $\bar{\Omega}$ driven by the Wiener process W and with coefficients b and σ , direction of reflection along γ and initial condition $x \in \bar{\Omega}_0$.*

We prove Theorem 2.6 by completing the following steps. First, in Lemma 4.3, we use a penalty method to prove existence of solutions to the Skorohod problem for smooth functions. In Lemma 4.4, we then derive a compactness estimate for solutions to the Skorohod problem. Based on the compactness estimate, we are, in Theorem 4.5, able to generalize the existence result for the Skorohod problem to all continuous functions. Finally, in Section 5, we use two classes of test functions and the existence result for the Skorohod problem to obtain existence and uniqueness of strong solutions to SDEs with oblique reflection at the boundary of a bounded, time-dependent domain. Note that we are able to obtain uniqueness of the reflected SDE although the solution to the corresponding Skorohod problem need not be unique.

2.3. Statement of main results for PDEs. To state and prove our results for PDEs we introduce some more notation. Let

$$(2.18) \quad \Omega^\circ = \Omega' \cap ((0, T) \times \mathbb{R}^n) \quad \text{and} \quad \tilde{\Omega} = \bar{\Omega}' \cap ((0, T) \times \mathbb{R}^n),$$

where Ω' is as in (2.1). We also put $\partial\Omega = \tilde{\Omega} \setminus \Omega^\circ$. We consider fully nonlinear parabolic PDEs of the form

$$(2.19) \quad u_t + F(t, x, u, Du, D^2u) = 0 \quad \text{in } \Omega^\circ.$$

Here F is a given real function on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$, where \mathbb{S}^n denotes the space of $n \times n$ real symmetric matrices equipped with the positive semi-definite ordering; that is, for $X, Y \in \mathbb{S}^n$, we write $X \leq Y$ if $\langle (X - Y)\xi, \xi \rangle \leq 0$ for all $\xi \in \mathbb{R}^n$. Moreover, u represents a real function in Ω° and Du and D^2u denote the gradient and Hessian matrix, respectively, of u with respect to the spatial variables. On the boundary we impose the oblique derivative condition to the unknown u

$$(2.20) \quad \frac{\partial u}{\partial \tilde{\gamma}} + f(t, x, u(t, x)) = 0 \quad \text{on } \partial\Omega,$$

where f is a real valued function on $\bar{\partial\Omega} \times \mathbb{R}$ and $\tilde{\gamma}(t, \cdot)$ is the vector field on \mathbb{R}^n , oblique to $\partial\Omega_t$, introduced in (2.5) and (2.6).

Regarding the function F , we make the following assumptions.

$$(2.21) \quad F \in C(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n).$$

For some $\lambda \in \mathbb{R}$ and each $(t, x, p, A) \in \bar{\Omega} \times \mathbb{R}^n \times \mathbb{S}^n$ the function

$$(2.22) \quad r \rightarrow F(t, x, r, p, A) - \lambda r \text{ is nondecreasing on } \mathbb{R}.$$

There is a function $m_1 \in C([0, \infty))$ satisfying $m_1(0) = 0$ for which

$$(2.23) \quad F(t, y, r, p, -Y) - F(t, x, r, p, X) \leq m_1(|x - y|(|p| + 1) + \alpha|x - y|^2) \\ \text{if} \quad -\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

for all $\alpha \geq 1$, $(t, x), (t, y) \in \overline{\Omega}$, $r \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $X, Y \in \mathbb{S}^n$, where I denotes the unit matrix of size $n \times n$. There is a neighborhood U of $\partial\Omega$ in $\overline{\Omega}$ and a function $m_2 \in C([0, \infty))$ satisfying $m_2(0) = 0$ for which

$$(2.24) \quad |F(t, x, r, p, X) - F(t, x, r, q, Y)| \leq m_2(|p - q| + \|X - Y\|)$$

for $(t, x) \in U$, $r \in \mathbb{R}$, $p, q \in \mathbb{R}^n$ and $X, Y \in \mathbb{S}^n$. Regarding the function f we assume that

$$(2.25) \quad f(t, x, r) \in C(\overline{\partial\Omega} \times \mathbb{R}),$$

and that for each $(t, x) \in \overline{\partial\Omega}$ the function

$$(2.26) \quad r \rightarrow f(t, x, r) \text{ is nondecreasing on } \mathbb{R}.$$

We remark that assumptions (2.21) and (2.23) imply the degenerate ellipticity

$$(2.27) \quad F(t, x, r, p, A + B) \leq F(t, x, r, p, A) \quad \text{if } B \geq 0$$

for $(t, x) \in \overline{\Omega}$, $r \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $A, B \in \mathbb{S}^n$, see Remark 3.4 in [8] for a proof. To handle the strong degeneracy allowed, we will adapt the notion of viscosity solutions [8], which we recall for problem (2.19)-(2.20) in Section 6.

Our main results concerning parabolic PDE's in time-dependent domains are summarized in the following theorem.

Theorem 2.7. *Let Ω° be a time-dependent domain satisfying (2.18) and assume that (2.5)-(2.7) and (2.21)-(2.26) hold. Let $u, -v \in USC(\tilde{\Omega} \cup \overline{\Omega}_0)$ be a viscosity subsolution and a viscosity supersolution, respectively, of problem (2.19)-(2.20) in Ω° . If $u(0, x) \leq v(0, x)$ for all $x \in \overline{\Omega}_0$, then*

$$u \leq v \quad \text{in } \tilde{\Omega}.$$

Moreover, if such u and v exist with $u = v = g$ on $\overline{\Omega}_0$ for some $g \in C(\overline{\Omega}_0)$, then there exists a unique viscosity solution to the initial value problem given by (2.19)-(2.20) and

$$(2.28) \quad u(0, x) = g(x) \quad \text{for } x \in \overline{\Omega}_0.$$

The comparison principle in Theorem 2.7 is proved in Section 6.1 using two of the test functions constructed in Section 3 together with nowadays standard techniques from the theory of viscosity solutions for fully nonlinear PDEs as described in [8]. Our proof uses ideas from the corresponding elliptic result given in [9]. To prove the existence part we use Perron's method. Uniqueness is immediate from the formulation of Theorem 2.7.

3. CONSTRUCTION OF TEST FUNCTIONS

In this section we show how the class of test function constructed in [9] for time-independent domains can be generalized to a similar class of test function valid for time-dependent domains. We describe the construction with sufficient detail and refer the reader to [9] for those parts of the construction that are identical in time-dependent and time-independent domains. We start by stating the result of Lemma 4.4 in [9] which asserts that for any $\theta \in (0, 1)$ there exists a function $g \in \mathcal{C}(\mathbb{R}^{2n}, \mathbb{R})$ and positive constants χ, C such that

$$(3.1) \quad g \in C^1(\mathbb{R}^{2n}, \mathbb{R}) \cap C^2(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}), \mathbb{R}),$$

$$(3.2) \quad g(\xi, p) \geq \chi |p|^2, \quad \text{for } \xi \in S(0, 1), p \in \mathbb{R}^n,$$

$$(3.3) \quad g(\xi, 0) = 0, \quad \text{for } \xi \in \mathbb{R}^n,$$

$$(3.4) \quad \langle D_p g(\xi, p), \xi \rangle \geq 0, \quad \text{if } \xi \in S(0, 1), p \in \mathbb{R}^n \text{ and } \langle p, \xi \rangle \geq -\theta |p|,$$

$$(3.5) \quad \langle D_p g(\xi, p), \xi \rangle \leq 0, \quad \text{if } \xi \in S(0, 1), p \in \mathbb{R}^n \text{ and } \langle p, \xi \rangle \leq \theta |p|,$$

and

$$(3.6) \quad |D_\xi g(\xi, p)| \leq C |p|^2, \quad \|D_\xi^2 g(\xi, p)\| \leq C |p|^2,$$

$$(3.7) \quad |D_p g(\xi, p)| \leq C |p|, \quad \|D_\xi D_p g(\xi, p)\| \leq C |p|,$$

$$(3.8) \quad \|D_p^2 g(\xi, p)\| \leq C,$$

for $\xi \in S(0, 1)$, $p \in \mathbb{R}^n \setminus \{0\}$. The test function provided by the following lemma will be used to assert relative compactness of solutions to the Skorohod problem in Lemma 4.4 below.

Lemma 3.1. *For any $\theta \in (0, 1)$, there exists a function $h \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^{2n}, \mathbb{R})$ and positive constants χ, C such that*

$$(3.9) \quad h(t, x, p) \geq \chi |p|^2,$$

$$(3.10) \quad h(t, x, 0) = 1,$$

$$(3.11) \quad \langle D_p h(t, x, p), \gamma(t, x) \rangle \geq 0, \quad \text{if } \langle p, \gamma(t, x) \rangle \geq -\theta |p|,$$

$$(3.12) \quad \langle D_p h(t, x, p), \gamma(t, x) \rangle \leq 0, \quad \text{if } \langle p, \gamma(t, x) \rangle \leq \theta |p|,$$

$$(3.13) \quad |D_t h(t, x, p)| \leq C |p|^2, \quad |D_x h(t, x, p)| \leq C |p|^2, \quad \|D_x^2 h(t, x, p)\| \leq C |p|^2,$$

$$(3.14) \quad |D_p h(t, x, p)| \leq C |p|, \quad \|D_x D_p h(t, x, p)\| \leq C |p|,$$

$$(3.15) \quad \|D_p^2 h(t, x, p)\| \leq C,$$

for $t \in [0, T]$, $p \in \mathbb{R}^n$, x belongs to a compactly supported subset of \mathbb{R}^n .

Proof. Let $\nu \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ be such that $\nu(t) = t$ for $t \geq 2$, $\nu(t) = 1$ for $t \leq 1/2$, $\nu'(t) \geq 0$ and $\nu(t) \geq t$ for all $t \in \mathbb{R}$. Let $\theta \in (0, 1)$ be given and choose $g \in \mathcal{C}(\mathbb{R}^{2n}, \mathbb{R})$ satisfying (3.1)-(3.8) and define

$$h(t, x, p) = \nu(g(\gamma(t, x), p)).$$

The regularity of h follows easily from the regularity of g and ν and (3.2)-(3.3). It is quite straightforward to deduce properties (3.9)-(3.15) from (3.1)-(3.8) and we limit the proof to two examples, which are not fully covered in [9]. For x belonging to a compactly supported subset of \mathbb{R}^n , we have

$$|D_t h(t, x, p)| = |\nu'(g(t, \gamma(t, x), p))| |D_\xi g(t, \gamma(t, x), p)| \left| \frac{\partial \gamma}{\partial t} \right| \leq C |p|^2,$$

by (3.6) and the regularity of ν and γ . Moreover,

$$\begin{aligned} \|D_x^2 h(t, x, p)\| &\leq |\nu''(g(t, \gamma(t, x), p))| |D_\xi g(t, \gamma(t, x), p)|^2 \left| \frac{\partial \gamma}{\partial x} \right|^2 \\ &\quad + |\nu'(g(t, \gamma(t, x), p))| |D_\xi^2 g(t, \gamma(t, x), p)| \left| \frac{\partial \gamma}{\partial x} \right|^2 \\ &\quad + |\nu'(g(t, \gamma(t, x), p))| |D_\xi g(t, \gamma(t, x), p)| \left| \frac{\partial^2 \gamma}{\partial x^2} \right|. \end{aligned}$$

Since ν'' is zero unless $2 \geq g(\gamma(t, x), p) \geq \chi |p|^2$, the first term, which is of order $C |p|^4$, only contributes for small $|p|^2$ and can thus be bounded from above by $C |p|^2$. By (3.6), the two latter terms are also bounded from above by $C |p|^2$, but for the last term the bound only holds for x belonging to a compactly supported subset of \mathbb{R}^n . \square

We conclude this section by verifying the existence of two additional test functions that will be used to prove Theorem 2.7 and Theorem 5.1. The test function in Lemma 3.2 is derived from the test function in Lemma 3.1 and the test function in Lemma 3.3 is derived from the solution to an appropriate Cauchy problem.

Lemma 3.2. *For any $\theta \in (0, 1)$, there exists a family $\{w_\varepsilon\}_{\varepsilon>0}$ of functions $w_\varepsilon \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^{2n}, \mathbb{R})$ and positive constants χ, C (independent of ε) such that*

$$(3.16) \quad w_\varepsilon(t, x, y) \geq \chi \frac{|x - y|^2}{\varepsilon},$$

$$(3.17) \quad w_\varepsilon(t, x, y) \leq C \left(\varepsilon + \frac{|x - y|^2}{\varepsilon} \right),$$

$$(3.18) \quad \langle D_x w_\varepsilon(t, x, y), \gamma(t, x) \rangle \leq C \frac{|x - y|^2}{\varepsilon}, \quad \text{if } \langle y - x, \gamma(t, x) \rangle \geq -\theta |x - y|,$$

$$(3.19) \quad \langle D_y w_\varepsilon(t, x, y), \gamma(t, x) \rangle \leq 0, \quad \text{if } \langle x - y, \gamma(t, x) \rangle \geq -\theta |x - y|,$$

$$(3.20) \quad \langle D_y w_\varepsilon(t, x, y), \gamma(t, y) \rangle \leq C \frac{|x - y|^2}{\varepsilon}, \quad \text{if } \langle x - y, \gamma(t, y) \rangle \geq -\theta |x - y|,$$

$$(3.21) \quad |D_t w_\varepsilon(t, x, y)| \leq C \frac{|x - y|^2}{\varepsilon},$$

$$(3.22) \quad |D_y w_\varepsilon(t, x, y)| \leq C \frac{|x - y|}{\varepsilon}, \quad |D_x w_\varepsilon(t, x, y) + D_y w_\varepsilon(t, x, y)| \leq C \frac{|x - y|^2}{\varepsilon},$$

$$(3.23) \quad D^2 w_\varepsilon(t, x, y) \leq \frac{C}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \frac{C|x - y|^2}{\varepsilon} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

for $t \in [0, T]$, x, y belong to a compactly supported subset of \mathbb{R}^n .

Proof. Let $\theta \in (0, 1)$ be given and choose $h \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^{2n}, \mathbb{R})$ satisfying (3.9)-(3.15). For all $\varepsilon > 0$, we define the function w_ε as

$$w_\varepsilon(t, x, y) = \varepsilon h\left(t, x, \frac{x - y}{\varepsilon}\right).$$

Property (3.16) follows easily from (3.9) and property (3.17) was verified in Remark 3.3 in [11]. Moreover, properties (3.18)-(3.19) and (3.22)-(3.23) were verified in the proof of Theorem 4.1 in [9]. To prove (3.20), we note that if $\langle x - y, \gamma(t, y) \rangle \geq -\theta |x - y|$, then by (3.11), $\langle D_p h(t, y, p), \gamma(t, y) \rangle \geq 0$ with $p = (x - y)/\varepsilon$. Hence, with the aid of (3.14), we obtain

$$\begin{aligned} & \langle D_y w_\varepsilon(t, x, y), \gamma(t, y) \rangle \\ &= - \left\langle D_p h\left(t, x, \frac{x - y}{\varepsilon}\right), \gamma(t, y) \right\rangle \\ &= - \left\langle D_p h\left(t, y, \frac{x - y}{\varepsilon}\right), \gamma(t, y) \right\rangle \\ & \quad + \left\langle D_p h\left(t, y, \frac{x - y}{\varepsilon}\right) - D_p h\left(t, x, \frac{x - y}{\varepsilon}\right), \gamma(t, y) \right\rangle \\ &\leq 0 + \left| D_p h\left(t, y, \frac{x - y}{\varepsilon}\right) - D_p h\left(t, x, \frac{x - y}{\varepsilon}\right) \right| |\gamma(t, y)| \\ &\leq \left\| D_x D_p h\left(t, x, \frac{x - y}{\varepsilon}\right) \right\| |x - y| \leq C \left| \frac{x - y}{\varepsilon} \right| |x - y| = C \frac{|x - y|^2}{\varepsilon}. \end{aligned}$$

Finally, (3.21) is a simple consequence of (3.13). \square

Lemma 3.3. *There exists a nonnegative function $\alpha \in \mathcal{C}^{1,2}(\overline{\Omega}, \mathbb{R})$, which satisfies*

$$(3.24) \quad \langle D_x \alpha(t, x), \gamma(t, x) \rangle \geq 1,$$

for $x \in \partial\Omega_t$, $t \in [0, T]$. Moreover, the support of α can be assumed to lie in the neighbourhood U defined in (2.24).

Proof. Fix $s \in [0, T]$ and $z \in \partial\Omega_s$ and define $H_{s,z}$ as the hyperplane

$$H_{s,z} = \{x \in \mathbb{R}^n : \langle x - z, \gamma(s, z) \rangle = 0\}.$$

Given a function $u_0 \in \mathcal{C}^2(H_{s,z}, \mathbb{R})$, such that $u_0(z) = 1$, $u_0 \geq 0$ and $\text{supp } u_0 \subset B(z, \delta^2/4) \cap H_{s,z}$, we can use the method of characteristics to solve the Cauchy problem

$$\begin{aligned} \langle D_x u_{(t)}(x), \gamma(t, x) \rangle &= 0, \\ u_{(t)}|_{H_{s,z}} &= u_0. \end{aligned}$$

Choosing the positive constants δ and η sufficiently small, the Cauchy problem above has, for all $t \in [s - \eta, s + \eta]$, a solution $u_{(t)} \in \mathcal{C}^2(B(z, \delta), \mathbb{R})$ satisfying $u_{(t)} \geq 0$. Based on the continuity of γ and the restriction on the support of u_0 , we may also assume that

$$\text{supp } u_{(t)} \subset \bigcup_{\zeta \in \mathbb{R}} B(z - \zeta \gamma(s, z), \delta^2/3) \cap B(z, \delta).$$

Next, we define the combined function

$$u(t, x) = u_{(t)}(x),$$

and we claim for now that $u \in \mathcal{C}^{1,2}([s - \eta, s + \eta] \times B(z, \delta), \mathbb{R})$ and postpone the proof of this claim to the end of the proof of the lemma. By the exterior and interior cone conditions, we can, for sufficiently small δ , find $\varepsilon > 0$ such that

$$\begin{aligned} &\bigcup_{\zeta \in \mathbb{R}} B(z - \zeta \gamma(s, z), \delta^2/3) \cap B(z, \delta) \setminus B(z, \delta - 2\varepsilon) \\ &\subset \bigcup_{\zeta \in \mathbb{R}} B(z - \zeta \gamma(s, z), \zeta \delta) \cap B(z, \delta) \subset \Omega_s^c, \end{aligned}$$

and hence

$$\partial\Omega_s \cap \text{supp } u_{(s)} \setminus B(z, \delta - 2\varepsilon) = \emptyset.$$

Then, by (2.8), it follows that if η also satisfies the constraint $\eta < (\varepsilon/K)^2$, then

$$(3.25) \quad \partial\Omega_t \cap (\text{supp } u_{(t)} \setminus B(z, \delta - \varepsilon)) = \emptyset.$$

for all $t \in [s - \eta, s + \eta]$.

Now, choose a function $\xi \in \mathcal{C}_0^{1,2}([s - \eta, s + \eta] \times B(z, \delta), \mathbb{R})$ so that $\xi(t, x) = 1$ for $t \in [s - \eta, s + \eta]$, $x \in B(z, \delta - \varepsilon)$ and $\xi \geq 0$, and set

$$v_{s,z}(t, x) = u(t, x) \xi(t, x),$$

Then $v_{s,z} \in \mathcal{C}_0^{1,2}([s - \eta, s + \eta] \times B(z, \delta), \mathbb{R})$ satisfies $v_{s,z} \geq 0$. By (3.25) and the construction of u and ξ , we obtain

$$\langle D_x v_{s,z}(t, x), \gamma(t, x) \rangle = 0 \text{ for } x \in B(z, \delta) \cap \partial\Omega_t, t \in [s - \eta, s + \eta].$$

Define $w_{s,z} \in \mathcal{C}^{1,2}([s - \eta, s + \eta] \times B(z, \delta), \mathbb{R})$ by

$$w_{s,z}(t, x) = \langle x - z, \gamma(s, z) \rangle + M,$$

where M is large enough so that $w_{s,z} \geq 0$. Using the continuity of γ , we can find δ and η such that $\langle \gamma(s, z), \gamma(t, x) \rangle \geq 0$ for all $(t, x) \in [s - \eta, s + \eta] \times B(z, \delta)$. Setting

$$g_{s,z}(t, x) = v_{s,z}(t, x) w_{s,z}(t, x),$$

we find that $g_{s,z} \in \mathcal{C}_0^{1,2}([s-\eta, s+\eta] \times B(z, \delta), \mathbb{R})$ satisfies $g_{s,z} \geq 0$. Moreover, using $|\gamma(t, x)| = 1$, we have

$$\begin{aligned} \langle D_x g_{s,z}(s, z), \gamma(s, z) \rangle &= v_{s,z}(s, z) \langle D_x w_{s,z}(s, z), \gamma(s, z) \rangle \\ &\quad + w_{s,z}(s, z) \langle D_x v_{s,z}(s, z), \gamma(s, z) \rangle \\ &= u(s, z) \xi(s, z) |\gamma(s, z)|^2 = 1, \end{aligned}$$

and a similar calculation shows that

$$\begin{aligned} \langle D_x g_{s,z}(t, x), \gamma(t, x) \rangle &= v_{s,z}(t, x) \langle D_x w_{s,z}(t, x), \gamma(t, x) \rangle \\ &\quad + w_{s,z}(t, x) \langle D_x v_{s,z}(t, x), \gamma(t, x) \rangle \\ &= v_{s,z}(t, x) \langle \gamma(s, z), \gamma(t, x) \rangle \geq 0, \end{aligned}$$

for $x \in B(z, \delta) \cap \partial\Omega_t$, $t \in [s-\eta, s+\eta]$. Now, using the same compactness argument as in [9], we conclude the existence of a nonnegative function $\alpha \in \mathcal{C}^{1,2}(\overline{\Omega}, \mathbb{R})$, which satisfies $\langle D_x \alpha(t, x), \gamma(t, x) \rangle \geq 1$ for $x \in \partial\Omega_t$, $t \in [0, T]$. Moreover, by the above construction, we can assume that the support of α lies within the neighbourhood U defined in (2.24).

It remains to prove the proposed regularity $u \in \mathcal{C}^{1,2}([s-\eta, s+\eta] \times B(z, \delta), \mathbb{R})$. The regularity in the spatial variables follows directly by construction, so it remains to show that u is continuously differentiable in the time variable. Let $x \in B(z, \delta)$ and let t and $t+h$ belong to $[s-\eta, s+\eta]$. Denote by $y(t, \cdot)$ and $y(t+h, \cdot)$ the characteristic curves through x for the vector fields $\gamma(t, \cdot)$ and $\gamma(t+h, \cdot)$, respectively, so that

$$\begin{aligned} \frac{\partial y}{\partial r}(t, r) &= \pm \gamma(t, y(t, r)), \\ y(t, 0) &= x, \end{aligned}$$

and analogously for $y(t+h, \cdot)$. Choose the sign in the parametrization of $y(t, \cdot)$ so that there exists some $r(t) > 0$ such that $y(t, r(t)) = z(t) \in H_{s,z}$. Choosing the same sign in the parametrization of $y(t+h, \cdot)$ asserts the existence of some $r(t+h) > 0$ such that $y(t+h, r(t+h)) = z(t+h) \in H_{s,z}$. Without lack of generality, we assume the sign above to be positive. Since $u(t, x) = u_0(z(t))$, where u_0 is continuously differentiable, it remains to show that the function z is continuously differentiable. We first show that $y(\cdot, r)$ is continuously differentiable. Differentiating the Cauchy problem formally with respect to the time variable and introducing the function $\psi(t, r) = \frac{\partial y}{\partial t}(t, r)$, we obtain

$$\begin{aligned} \frac{\partial}{\partial r} \psi(t, r) &= \frac{\partial \gamma}{\partial t}(t, y(t, r)) + \frac{\partial \gamma}{\partial y}(t, y(t, r)) \psi(t, r), \\ \psi(t, 0) &= 0. \end{aligned}$$

This Cauchy problem has a unique solution, which we will next show satisfies

$$\psi(t, r) = \lim_{h \rightarrow 0} \frac{y(t+h, r) - y(t, r)}{h},$$

so that ψ is in fact the time derivative of y (not just formally). Define

$$R(t, r, h) = \frac{y(t+h, r) - y(t, r)}{h} - \psi(t, r).$$

Now

$$\begin{aligned} R(t, r, h) &= \int_0^r \left(\frac{\gamma(t+h, y(t+h, u)) - \gamma(t, y(t, u))}{h} \right) du \\ &\quad - \int_0^r \left(\frac{\partial \gamma}{\partial t}(t, y(t, u)) + \frac{\partial \gamma}{\partial y}(t, y(t, u)) \psi(t, u) \right) du. \end{aligned}$$

By the mean value theorem

$$\begin{aligned} & \gamma(t+h, y(t+h, u)) - \gamma(t, y(t, u)) \\ &= \frac{\partial \gamma}{\partial t}(\bar{t}, y(t, u)) h + \frac{\partial \gamma}{\partial y}(t, \bar{y})(y(t+h, u) - y(t, u)), \end{aligned}$$

for some \bar{t} between t and $t+h$ and some \bar{y} between $y(t, u)$ and $y(t+h, u)$. Hence

$$\begin{aligned} & R(t, r, h) \\ &= \int_0^r \left(\frac{\partial \gamma}{\partial t}(\bar{t}, y(t, u)) - \frac{\partial \gamma}{\partial t}(t, y(t, u)) \right) du \\ & \quad + \int_0^r \left(\frac{\partial \gamma}{\partial y}(t, \bar{y}) \frac{y(t+h, u) - y(t, u)}{h} - \frac{\partial \gamma}{\partial y}(t, y(t, u)) \psi(t, u) \right) du, \end{aligned}$$

where the second term on the right hand side can be rewritten as

$$\int_0^r \left(\frac{\partial \gamma}{\partial y}(t, \bar{y}) R(t, u, h) + \left(\frac{\partial \gamma}{\partial y}(t, \bar{y}) - \frac{\partial \gamma}{\partial y}(t, y(t, u)) \right) \psi(t, u) \right) du.$$

By Gronwall's inequality, we obtain

$$\begin{aligned} |R(t, r, h)| &\leq C \left(\int_0^r \left| \frac{\partial \gamma}{\partial t}(\bar{t}, y(t, u)) - \frac{\partial \gamma}{\partial t}(t, y(t, u)) \right| \right. \\ & \quad \left. + \left| \frac{\partial \gamma}{\partial y}(t, \bar{y}) - \frac{\partial \gamma}{\partial y}(t, y(t, u)) \right| |\psi(t, u)| \right) du, \end{aligned}$$

for some positive constant C . By the continuity of the time and space derivatives of γ , the differences in the integrand vanish as $h \rightarrow 0$. Moreover $|\psi(t, u)|$ exists and is bounded, so we may conclude that $\lim_{h \rightarrow 0} R(t, r, h) = 0$. By the uniform continuity of the derivatives of γ , the convergence is uniform in t , proving that $y(\cdot, r)$ is continuously differentiable.

Now, by the mean value theorem,

$$\begin{aligned} & z(t+h) - z(t) \\ &= y(t+h, r(t+h)) - y(t, r(t)) \\ &= y(t+h, r(t+h)) - y(t+h, r(t)) + y(t+h, r(t)) - y(t, r(t)) \\ &= \frac{\partial y}{\partial r}(t+h, \bar{r})(r(t+h) - r(t)) + \frac{\partial y}{\partial t}(\bar{t}, r(t)) h, \end{aligned}$$

for some \bar{r} between $r(t)$ and $r(t+h)$ and some \bar{t} between t and $t+h$. Since both $y(t+h, r(t+h))$ and $y(t, r(t))$ belong to $H_{s,z}$, the difference $y(t+h, r(t+h)) - y(t, r(t))$ is orthogonal to $\gamma(s, z)$ and

$$\left\langle \frac{\partial y}{\partial r}(t+h, \bar{r})(r(t+h) - r(t)) + \frac{\partial y}{\partial t}(\bar{t}, r(t)) h, \gamma(s, z) \right\rangle = 0.$$

Rearranging the terms, and letting $h \rightarrow 0$, we obtain

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{r(t+h) - r(t)}{h} &= - \lim_{h \rightarrow 0} \frac{\left\langle \frac{\partial y}{\partial t}(\bar{t}, r(t)), \gamma(s, z) \right\rangle}{\left\langle \frac{\partial y}{\partial r}(t+h, \bar{r}), \gamma(s, z) \right\rangle} \\ &= - \frac{\left\langle \frac{\partial y}{\partial t}(t, r(t)), \gamma(s, z) \right\rangle}{\left\langle \frac{\partial y}{\partial r}(t, r(t)), \gamma(s, z) \right\rangle}, \end{aligned}$$

by the continuity of the derivatives of y . As

$$\left\langle \frac{\partial y}{\partial r}(t, r(t)), \gamma(s, z) \right\rangle = \langle \gamma(t, y(t, r(t))), \gamma(s, z) \rangle > 0,$$

for sufficiently small δ and η , we conclude, since y is continuously differentiable, that r is a continuously differentiable function. Hence

$$\lim_{h \rightarrow 0} \frac{z(t+h) - z(t)}{h} = \frac{\partial y}{\partial r}(t, r(t)) r'(t) + \frac{\partial y}{\partial t}(t, r(t)),$$

where the right hand side is a continuous function. This proves that z is continuously differentiable and, hence, that $u \in \mathcal{C}^{1,2}([s-\eta, s+\eta] \times B(z, \delta), \mathbb{R})$. \square

Note that we only need Lipschitz continuity in time for the test function α in Lemma 3.3 to prove Theorem 5.1. To prove Theorem 2.7 we need $\alpha \in \mathcal{C}^{1,2}(\overline{\Omega})$.

4. SOLUTIONS TO THE SKOROHOD PROBLEM

In this section we prove existence of solutions to the Skorohod problem under the assumptions in Section 2.1. This result could be achieved using the methods in [19], but as we here assume more regularity on the direction of reflection and the temporal variation of the domain compared to the setting in [19] (and this is essential for the other sections of this article), we follow a more direct approach using a penalty method. We first note that, mimicking the proof of Lemma 4.1 in [11], we can prove the following result.

Lemma 4.1. *Under the assumptions in Section 2.1, there is a constant $\mu > 0$ such that, for every $t \in [0, T]$, there exists a neighbourhood U_t of $\partial\Omega_t$ such that*

$$(4.1) \quad \langle D_x d(t, x), \gamma(t, x) \rangle \leq -\mu, \quad \text{for a.e. } x \in U_t \setminus \overline{\Omega}_t.$$

As (4.1) holds only for almost every point in a neighbourhood of a non-smooth domain, we cannot apply (4.1) directly, but have to use the following mollifier approach. Based on the construction of the neighbourhoods $\{U_t\}_{t \in [0, T]}$ in Lemma 4.1 (see the proof of the corresponding lemma in [11] for details), there exists a constant $\beta > 0$ such that $B(x, 3\beta) \subset U_t$ for all $x \in \partial\Omega_t$, $t \in [0, T]$. Let $v(t, x) = (d(t, x))^2$ and let $\varphi_\beta \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ be a positive mollifier with support in $B(0, \beta)$. Define the spatial convolutions $v_\beta = v * \varphi_\beta$ and $d_\beta = d * \varphi_\beta$.

Lemma 4.2. *Under the assumptions in Section 2.1, there is, for sufficiently small β , a constant $\kappa > 0$ such that, for every $t \in [0, T]$, there exists a neighbourhood V_t of $\partial\Omega_t$ such that*

$$(4.2) \quad \langle D_x v_\beta(t, x), \gamma(t, x) \rangle \leq -\kappa d_\beta(t, x), \quad \text{for } x \in V_t \setminus \overline{\Omega}_t.$$

Proof. For all $x \in U_t \setminus \overline{\Omega}_t$ such that $B(x, \beta) \subset U_t$, we have

$$\begin{aligned} & \langle D_x v_\beta(t, x), \gamma(t, x) \rangle \\ &= \left\langle \int_{\mathbb{R}^n} D_x \varphi_\beta(x-y) v(t, y) dy, \gamma(t, x) \right\rangle \\ &= \left\langle \int_{\mathbb{R}^n} \varphi_\beta(x-y) D_y v(t, y) dy, \gamma(t, x) \right\rangle \\ &= \int_{\mathbb{R}^n} (\langle D_y v(t, y), \gamma(t, y) \rangle + \langle D_y v(t, y), \gamma(t, x) - \gamma(t, y) \rangle) \varphi_\beta(x-y) dy. \end{aligned}$$

The inner product in the second term can be bounded from above by $2d(t, y)L\beta$, where L is the Lipschitz coefficient of γ in spatial dimensions over the compact set $[0, T] \times \bigcup_{t \in [0, T]} \overline{U}_t$. By Lemma 4.1, we have for almost every $y \in U_t \setminus \overline{\Omega}_t$, $t \in [0, T]$,

$$\langle D_y v(t, y), \gamma(t, y) \rangle = 2d(t, y) \langle D_x d(t, y), \gamma(t, y) \rangle \leq -2\mu d(t, y).$$

Therefore

$$\begin{aligned} & \int_{\mathbb{R}^n} (\langle D_y v(t, y), \gamma(t, y) \rangle + \langle D_y v(t, y), \gamma(t, x) - \gamma(t, y) \rangle) \varphi_\beta(x - y) dy \\ & \leq 2(-\mu + L\beta) d_\beta(t, x) \leq -\kappa d_\beta(t, x), \end{aligned}$$

for sufficiently small $\beta > 0$. \square

It follows from the construction that the neighbourhoods $\{V_t\}_{t \in [0, T]}$ in (4.2) satisfies $B(x, 2\beta) \in V_t$ for all $x \in \partial\Omega_t$, $t \in [0, T]$. We next use a penalty method to verify the existence of a solution to the Skorohod problem for continuously differentiable functions. The following lemma is similar to, and generalizes, Theorem 2.1 in [17] and Lemma 4.5 in [11].

Lemma 4.3. *Let $\psi \in C^1([0, T], \mathbb{R}^n)$ with $\psi(0) \in \overline{\Omega}_0$. Then there exists a solution $(\phi, \lambda) \in \mathcal{H}^1([0, T], \mathbb{R}^n) \times \mathcal{H}^1([0, T], \mathbb{R}^n)$ to the Skorohod problem for (Ω, γ, ψ) .*

Proof. Choose $\varepsilon > 0$ and consider the ordinary differential equation

$$(4.3) \quad \phi'_\varepsilon(t) = \frac{1}{\varepsilon} d(t, \phi_\varepsilon(t)) \gamma(t, \phi_\varepsilon(t)) + \psi'(t), \quad \phi_\varepsilon(0) = \psi(0),$$

for $\phi_\varepsilon(t)$, which has a unique solution on $t \in [0, T]$. Choose a constant $\kappa > 0$ and a family of neighbourhoods $\{V_t\}_{t \in [0, T]}$ as in Lemma 4.2. Let $c \geq \beta^2 > 0$ be a constant and choose a function $\zeta \in C^\infty([0, \infty), [0, \infty))$ such that

$$\zeta(r) = \begin{cases} r, & \text{for } r \leq c, \\ 2c, & \text{for } r \geq 3c, \end{cases}$$

and $0 \leq \zeta'(r) \leq 1$ for all $r \in [0, \infty)$. Note that we may choose c so small that $v(t, x) = (d(t, x))^2 \leq 4c$ implies $x \in V_t \cup \overline{\Omega}_t$ for all $t \in [0, T]$. Moreover, we note that if $\phi_\varepsilon(t) \notin V_t \cup \overline{\Omega}_t$, then $v_\beta(t, \phi_\varepsilon(t)) \geq 3c$ and, as a consequence, $\zeta'(v_\beta(t, \phi_\varepsilon(t))) = 0$. We next define the function $V(t) = \zeta(v_\beta(t, \phi_\varepsilon(t)))$, for $t \in [0, T]$, and investigate its time derivative. Let D_t denote the weak derivative guaranteed by (2.7) and observe that

$$\begin{aligned} V'(t) &= \zeta'(v_\beta(t, \phi_\varepsilon(t))) (D_t v_\beta(t, \phi_\varepsilon(t)) + \langle D_x v_\beta(t, \phi_\varepsilon(t)), \phi'_\varepsilon(t) \rangle) \\ &= \zeta'(v_\beta(t, \phi_\varepsilon(t))) \left(D_t v_\beta(t, \phi_\varepsilon(t)) \right. \\ (4.4) \quad & \left. + \left\langle D_x v_\beta(t, \phi_\varepsilon(t)), \frac{1}{\varepsilon} d(t, \phi_\varepsilon(t)) \gamma(t, \phi_\varepsilon(t)) + \psi'(t) \right\rangle \right), \end{aligned}$$

as $\phi_\varepsilon(t)$ solves (4.3). From Lemma 4.2, we have

$$\begin{aligned} & \zeta'(v_\beta(t, \phi_\varepsilon(t))) \left\langle D_x v_\beta(t, \phi_\varepsilon(t)), \frac{1}{\varepsilon} d(t, \phi_\varepsilon(t)) \gamma(t, \phi_\varepsilon(t)) \right\rangle \\ & \leq -\frac{\kappa}{\varepsilon} \zeta'(v_\beta(t, \phi_\varepsilon(t))) d(t, \phi_\varepsilon(t)) d_\beta(t, \phi_\varepsilon(t)), \end{aligned}$$

for $\phi_\varepsilon(t) \in V_t \setminus \overline{\Omega}_t$ and for all other $\phi_\varepsilon(t)$ both sides vanish. Integrating the estimate for V' , we obtain, suppressing the s -dependence in ϕ_ε and ψ to simplify the notation

$$\begin{aligned} & \zeta(v_\beta(t, \phi_\varepsilon(t))) - \zeta(v_\beta(0, \phi_\varepsilon(0))) \\ & + \frac{\kappa}{\varepsilon} \int_0^t \zeta'(v_\beta(s, \phi_\varepsilon)) d(s, \phi_\varepsilon) d_\beta(s, \phi_\varepsilon) ds \\ & \leq \int_0^t \zeta'(v_\beta(s, \phi_\varepsilon)) (D_s v_\beta(s, \phi_\varepsilon) + \langle D_x v_\beta(s, \phi_\varepsilon), \psi' \rangle) ds \\ & \leq 2 \int_0^t \zeta'(v_\beta(s, \phi_\varepsilon)) d_\beta(s, \phi_\varepsilon) (|D_s d_\beta(s, \phi_\varepsilon)| + |D_x d_\beta(s, \phi_\varepsilon)| |\psi'|) ds, \end{aligned}$$

for $t \in [0, T]$. Letting β tend to zero, we obtain, as $v_\beta(0, \phi_\varepsilon(0)) \rightarrow v(0, \psi(0)) = 0$ and $|D_x d_\beta| \leq 1$,

$$\begin{aligned} & \zeta(v(t, \phi_\varepsilon(t))) + \frac{\kappa}{\varepsilon} \int_0^t \zeta'(v(s, \phi_\varepsilon)) v(s, \phi_\varepsilon) ds \\ & \leq 2 \int_0^t \zeta'(v(s, \phi_\varepsilon)) d(s, \phi_\varepsilon) (|D_s d| + |\psi'|) ds \\ & \leq 2 \left(\int_0^t \zeta'(v(s, \phi_\varepsilon)) v(s, \phi_\varepsilon) ds \right)^{1/2} \left(\int_0^t \zeta'(v(s, \phi_\varepsilon)) (|D_s d| + |\psi'|)^2 ds \right)^{1/2}. \end{aligned}$$

Both terms on the left hand side are positive and each of the terms are therefore bounded from above by the right hand side. Hence

$$\begin{aligned} \frac{\kappa}{\varepsilon} \left(\int_0^t \zeta'(v(s, \phi_\varepsilon)) v(s, \phi_\varepsilon) ds \right)^{1/2} & \leq 2 \left(\int_0^t \zeta'(v(s, \phi_\varepsilon)) (|D_s d| + |\psi'|)^2 ds \right)^{1/2} \\ & \leq 2\sqrt{2} \left(\int_0^t |\zeta'| (|D_s d|^2 + |\psi'|^2) ds \right)^{1/2} \\ & \leq C(T) < \infty, \end{aligned}$$

since $|\zeta'| \leq 1$, $\psi \in \mathcal{H}^1([0, T], \mathbb{R}^n)$ and d satisfies (2.7). Hence

$$\zeta(v(t, \phi_\varepsilon(t))) + \frac{\kappa}{\varepsilon} \int_0^t \zeta'(v(s, \phi_\varepsilon)) v(s, \phi_\varepsilon) ds \leq K(T) \varepsilon,$$

We may assume that $\varepsilon > 0$ has been chosen small enough that $v(t, \phi_\varepsilon(t)) \leq c$, for all $t \in [0, T]$. Then, by the definition of ζ ,

$$(4.5) \quad \frac{1}{\varepsilon} (d(t, \phi_\varepsilon(t)))^2 + \frac{\kappa}{\varepsilon^2} \int_0^t (d(s, \phi_\varepsilon(s)))^2 ds \leq K(T),$$

for $t \in [0, T]$. The remainder of the proof follows along the lines of the proof of Lemma 4.5 in [11], but we give the details for completeness. Relation (4.5) asserts that the sequences $\{l_\varepsilon\}_{\varepsilon>0}$ and $\{\lambda_\varepsilon\}_{\varepsilon>0}$, where

$$l_\varepsilon(t) = \frac{1}{\varepsilon} d(t, \phi_\varepsilon(t)), \quad \lambda_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t d(s, \phi_\varepsilon(s)) \gamma(s, \phi_\varepsilon(s)) ds$$

are bounded in $\mathcal{L}^2([0, T], \mathbb{R}^n)$ and $\mathcal{H}^1([0, T], \mathbb{R}^n)$ respectively. Thus, we may assume that l_ε and λ_ε converge weakly to $l \in \mathcal{L}^2([0, T], \mathbb{R}^n)$ and $\lambda \in \mathcal{H}^1([0, T], \mathbb{R}^n)$, respectively, as $\varepsilon \rightarrow 0$. Moreover, from (4.3) we conclude that ϕ_ε converges weakly to $\phi \in \mathcal{H}^1([0, T], \mathbb{R}^n)$ and that $\phi(t) = \psi(t) + \lambda(t)$, $\phi(0) = \psi(0)$. This proves (2.9) and, moreover, (2.10) holds due to (4.5). Since $\lambda'_\varepsilon(t) = l_\varepsilon(t) \gamma(t, \phi_\varepsilon(t))$, we have $\lambda'(t) = l(t) \gamma(t, \phi(t))$ and therefore

$$|\lambda(t)| = \int_0^t |\lambda'(s)| ds = \int_0^t l(s) ds, \quad \text{for all } t \in [0, T],$$

which proves (2.11). In addition,

$$\lambda(t) = \int_0^t l(s) \gamma(s, \phi(s)) ds = \int_0^t \gamma(s, \phi(s)) d|\lambda|(s), \quad \text{for all } t \in [0, T],$$

which proves (2.13). It remains to verify (2.12). Let $\tau = \{t \in [0, T] : \phi(t) \in \Omega_t\}$ and note that for each fixed $t \in \tau$, we have $l_\varepsilon(t) = 0$ for all sufficiently small ε . Hence $l(t) = 0$ on τ and

$$|\lambda(t)| = \int_0^t I_{\{\phi(s) \in \partial\Omega_s\}} l(s) ds = \int_0^t I_{\{\phi(s) \in \partial\Omega_s\}} d|\lambda|(s).$$

This completes the proof that $(\phi, \lambda) \in \mathcal{H}^1([0, T], \mathbb{R}^n) \times \mathcal{H}^1([0, T], \mathbb{R}^n)$ solves the Skorohod problem for (Ω, γ, ψ) . \square

The next step is to prove relative compactness of solutions to the Skorohod problem. We define the modulus of continuity of a function $f \in \mathcal{C}([0, T], \mathbb{R}^n)$ as $\|f\|_{s,t} = \sup_{s \leq t_1 \leq t_2 \leq t} |f(t_2) - f(t_1)|$ for $0 \leq s \leq t \leq T$.

Lemma 4.4. *Let A be a compact subset of $\mathcal{C}([0, T], \mathbb{R}^n)$. Then*

(i): *There exists a constant $L < \infty$ such that*

$$|\lambda|(T) < L,$$

for all solutions $(\psi + \lambda, \lambda)$ to the Skorohod problem for (Ω, γ, ψ) with $\psi \in A$.

(ii): *The set*

$$\{\phi : (\phi, \lambda) \text{ solves the Skorohod problem for } (\Omega, \gamma, \psi) \text{ with } \psi \in A\}$$

is relatively compact.

Proof. The proof of this lemma follows the lines of the proof of Lemma 4.7 in [11], but a number of minor changes must be carried out carefully to handle the time dependency of the domain. By the compactness of $\overline{\Omega}$ and the continuity of γ , there exists a constant $c > 0$ such that for every $t \in [0, T]$ and $x \in \overline{\Omega}_t$, there exists a vector $v(t, x)$ and a set $[t, t+c] \times B(x, c)$ such that $\langle \gamma(s, y), v(t, x) \rangle > c$ for all $(s, y) \in [t, t+c] \times B(x, c)$. Without lack of generality, we may assume that $c < \delta$, for the δ introduced in Remark 2.1. Let $\psi \in A$ be given and let (ϕ, λ) be any solution to the Skorohod problem for (Ω, γ, ψ) . Define T_1 to be smallest of T, c and $\inf \{t \in [0, T] : \phi(t) \notin B(\phi(0), c)\}$. Next define T_2 to be the smallest of $T, T_1 + c$ and $\inf \{t \in [T_1, T] : \phi(t) \notin B(\phi(T_1), c)\}$. Continuing in this fashion, we obtain a sequence $\{T_m\}_{m=1,2,\dots}$ of time instants. By construction, for all $s \in [T_{m-1}, T_m]$ we have $s \in [T_{m-1}, T_{m-1} + c]$ and $\phi(s) \in B(\phi(T_{m-1}), c)$, and hence $\langle \gamma(s, \phi(s)), v(T_{m-1}, \phi(T_{m-1})) \rangle > c$. Hence

$$\begin{aligned} & \langle \phi(T_m) - \phi(T_{m-1}), v(T_{m-1}, \phi(T_{m-1})) \rangle \\ & - \langle \psi(T_m) - \psi(T_{m-1}), v(T_{m-1}, \phi(T_{m-1})) \rangle \\ & = \int_{T_{m-1}}^{T_m} \langle \gamma(s, \phi(s)), v(T_{m-1}, \phi(T_{m-1})) \rangle d|\lambda|(s) \geq c(|\lambda|(T_m) - |\lambda|(T_{m-1})). \end{aligned}$$

Since A is compact, the set $\{\psi(t) : t \in [0, T], \psi \in A\}$ is bounded. Moreover, since $\overline{\Omega}$ is compact and $\phi(t) \in \overline{\Omega}_t$ for all $t \in [0, T]$, there exists a constant $M < \infty$ such that

$$|\lambda|(T_m) - |\lambda|(T_{m-1}) < M.$$

We next prove that there exists a positive constant R such that, for any $\psi \in A$,

$$(4.6) \quad \|\lambda\|_{T_{m-1}, \tau} \leq R \left(\|\psi\|_{T_{m-1}, \tau}^{1/2} + \|\psi\|_{T_{m-1}, \tau}^{3/2} + (\tau - T_{m-1})^{1/4} \right),$$

for $T_{m-1} \leq \tau \leq T_m$. As we are only interested in the behaviour during the time interval $[T_{m-1}, T_m]$, we simplify the notation by setting, without loss of generality, $T_{m-1} = 0$, $\phi(T_{m-1}) = x$, $\psi(T_{m-1}) = x$, $\lambda(T_{m-1}) = 0$ and $|\lambda|(T_{m-1}) = 0$. Let h be the function in Lemma 3.1 and let χ, C be the corresponding positive constants. Define $B_\varepsilon(t) = \varepsilon h(t, x, -\lambda(t)/\varepsilon)$ and $E(t) = e^{-(2|\lambda|(t)+t)C/\chi}$. Since $h(t, x, 0) = 1$,

we get

$$\begin{aligned}
B_\varepsilon(\tau) E(\tau) &= B_\varepsilon(0) E(0) + \int_0^\tau (E(u) dB_\varepsilon(u) + B_\varepsilon(u) dE(u)) \\
&= \varepsilon + \int_0^\tau E(u) dB_\varepsilon(u) - \frac{2C}{\chi} \int_0^\tau B_\varepsilon(u) E(u) d|\lambda|(u) \\
&\quad - \frac{C}{\chi} \int_0^\tau B_\varepsilon(u) E(u) du,
\end{aligned}$$

where the first integral can be rewritten as

$$\begin{aligned}
\int_0^\tau E(u) dB_\varepsilon(u) &= \int_0^\tau E(u) \varepsilon D_t h(u, x, -\lambda(u)/\varepsilon) du \\
&\quad - \int_0^\tau E(u) \langle D_p h(u, x, -\lambda(u)/\varepsilon), d\lambda(u) \rangle.
\end{aligned}$$

By (3.9) and (3.13), the integral involving $D_t h$ has the upper bound

$$\begin{aligned}
&\int_0^\tau E(u) \varepsilon D_t h(u, x, -\lambda(u)/\varepsilon) du \\
&\leq C\varepsilon \int_0^\tau E(u) |\lambda(u)/\varepsilon|^2 du = \frac{C}{\varepsilon} \int_0^\tau E(u) |\lambda(u)|^2 du \\
&\leq \frac{C}{\chi} \int_0^\tau E(u) B_\varepsilon(u) du.
\end{aligned}$$

Next, we would like to find an upper bound for the integral involving $D_p h$ using (3.11) in some appropriate way, but we have to be somewhat careful due to the temporal variation of the domain. Assume that $\phi(u) \in \partial\Omega_u$. If $x \notin \overline{\Omega}_u$, there exists at least one point $y_u \in \overline{\Omega}_u \cap B(x, c)$ such that $|x - y_u| = d(u, x)$. We have chosen $c < \delta$, so $\langle y_u - \phi(u), \gamma(u, \phi(u)) \rangle \geq -\theta |y_u - \phi(u)|$ holds by Remark 2.1 and, due to (3.11), we can conclude

$$I_1 := - \int_0^\tau E(u) \langle D_p h(u, \phi(u), (y_u - \phi(u))/\varepsilon), \gamma(u, \phi(u)) \rangle d|\lambda|(u) \leq 0,$$

since $d|\lambda|(u) = 0$ if $\phi(u) \notin \partial\Omega_u$. If $x \in \overline{\Omega}_u$, the above estimate holds with y_u replaced by x . The integral involving $D_p h$ can be decomposed into

$$- \int_0^\tau E(u) \langle D_p h(u, x, -\lambda(u)/\varepsilon), d\lambda(u) \rangle = I_1 + I_2 + I_3,$$

for I_1 as above and

$$I_2 := \int_0^\tau E(u) \langle D_p h(u, \phi(u), -\lambda(u)/\varepsilon) - D_p h(u, x, -\lambda(u)/\varepsilon), d\lambda(u) \rangle,$$

$$I_3 := \int_0^\tau E(u) \langle D_p h(u, \phi(u), (y_u - \phi(u))/\varepsilon) - D_p h(u, \phi(u), -\lambda(u)/\varepsilon), d\lambda(u) \rangle.$$

By (3.9) and (3.14)-(3.15), these integrals can be bounded from above by

$$\begin{aligned}
I_2 &\leq \frac{C}{\varepsilon} \int_0^\tau E(u) |\lambda(u)| |x - \phi(u)| d\lambda(u) \\
&\leq \frac{C}{\varepsilon} \int_0^\tau E(u) \left(|\lambda(u)|^2 + |x - \psi(u)| |\lambda(u)| \right) d\lambda(u) \\
&\leq \frac{2C}{\varepsilon} \int_0^\tau E(u) \left(|\lambda(u)|^2 + |x - \psi(u)|^2 \right) d\lambda(u) \\
&\leq \frac{2C}{\chi} \int_0^\tau E(u) B_\varepsilon(u) d\lambda(u) + \frac{2C}{\varepsilon} \int_0^\tau E(u) |x - \psi(u)|^2 d\lambda(u),
\end{aligned}$$

and

$$\begin{aligned}
I_3 &\leq \frac{C}{\varepsilon} \int_0^\tau E(u) |y_u - \phi(u) - (-\lambda(u))| d\lambda(u) \\
&= \frac{C}{\varepsilon} \int_0^\tau E(u) |y_u - \psi(u)| d\lambda(u) \\
&\leq \frac{C}{\varepsilon} \int_0^\tau E(u) (|x - \psi(u)| + |y_u - x|) d\lambda(u) \\
&\leq \frac{C}{\varepsilon} \int_0^\tau E(u) (|x - \psi(u)| + d(u, x)) d\lambda(u).
\end{aligned}$$

Collecting all the terms, we obtain

$$B_\varepsilon(\tau) E(\tau) \leq \varepsilon + \frac{C}{\varepsilon} \int_0^\tau E(u) \left(|x - \psi(u)| + 2|x - \psi(u)|^2 + d(u, x) \right) d|\lambda|(u),$$

which implies

$$B_\varepsilon(\tau) \leq \left(\frac{2C}{\varepsilon} \int_0^\tau E(u) \left(\|\psi\|_{0,\tau} + \|\psi\|_{0,\tau}^2 + K\sqrt{\tau} \right) d|\lambda|(u) + \varepsilon \right) e^{(2|\lambda|(\tau)+\tau)C/\chi},$$

where $K > 0$ is the constant from Remark 2.2. Now

$$\int_0^\tau E(u) d|\lambda|(u) \leq \int_0^\tau e^{-2C|\lambda|(u)/\chi} d|\lambda|(u) \leq \frac{\chi}{2C},$$

so

$$B_\varepsilon(\tau) \leq \left(\frac{\chi}{\varepsilon} \left(\|\psi\|_{0,\tau} + \|\psi\|_{0,\tau}^2 + K\sqrt{\tau} \right) + \varepsilon \right) e^{(2|\lambda|(\tau)+\tau)C/\chi}.$$

Another application of (3.9) gives

$$\begin{aligned}
|\lambda(\tau)| &\leq \frac{1}{2} \left(\varepsilon + \frac{1}{\varepsilon} |\lambda(\tau)|^2 \right) \leq \frac{\varepsilon}{2} + \frac{B_\varepsilon(\tau)}{2\chi} \\
&\leq \frac{\varepsilon}{2} + \left(\frac{1}{2\varepsilon} \left(\|\psi\|_{0,\tau} + \|\psi\|_{0,\tau}^2 + K\sqrt{\tau} \right) + \frac{\varepsilon}{2\chi} \right) e^{(2M+T)C/\chi}.
\end{aligned}$$

Set $\varepsilon = \max \left\{ \|\psi\|_{0,\tau}^{1/2}, \tau^{1/4} \right\}$ so that $\varepsilon \leq \|\psi\|_{0,\tau}^{1/2} + \tau^{1/4}$, $1/\varepsilon \leq \|\psi\|_{0,\tau}^{-1/2}$ and $1/\varepsilon \leq \tau^{-1/4}$. Then (4.6) follows immediately from the above inequality. By (4.6) and the compactness of A , there exists an $\varepsilon > 0$ such that

$$\max \left\{ \|\psi\|_{T_{m-1}, T_{m-1}+\varepsilon}, \|\lambda\|_{T_{m-1}, T_{m-1}+\varepsilon} \right\} \leq c/3,$$

which implies $\|\phi\|_{T_{m-1}, T_{m-1}+\varepsilon} \leq 2c/3$. The definition of $\{T_m\}$ then implies that $T_m - T_{m-1} \geq \min \{\varepsilon, c\}$. This proves (i) with $L = MT / \min \{\varepsilon, c\}$. Part (ii) follows from (4.6) and the bound $T_m - T_{m-1} \geq \min \{\varepsilon, c\}$. \square

Equipped with the results above, we are now ready to state and prove the existence result for solutions to the Skorohod problem.

Theorem 4.5. *Let $\psi \in \mathcal{C}([0, T], \mathbb{R}^n)$ with $\psi(0) \in \overline{\Omega}_0$. Then there exists a solution (ϕ, λ) to the Skorohod problem for (Ω, γ, ψ) .*

Proof. The proof is very similar to the proof of Theorem 4.8 in [11], but for completeness we sketch the proof and refer to [11] for the remaining details. Let $\psi_n \in \mathcal{C}^1([0, T], \mathbb{R}^n)$ form a sequence of functions converging uniformly to ψ . According to Lemma 4.3, there exists a solution (ϕ_n, λ_n) to the Skorohod problem for (Ω, γ, ψ_n) . By Lemma 4.4, we may assume that the sequence $\{\lambda_n\}_{n=1}^\infty$ is equibounded and equicontinuous, that is

$$\begin{aligned}
\sup_n |\lambda_n|(T) &\leq L < \infty, \\
\lim_{|s-t| \rightarrow 0} \sup_n |\lambda_n(s) - \lambda_n(t)| &= 0.
\end{aligned}$$

The Arzela-Ascoli theorem asserts the existence of a function $\lambda \in \mathcal{C}([0, T], \mathbb{R}^n)$ such that $\{\lambda_n\}$ converges uniformly to λ . Clearly $|\lambda|(T) \leq L$. Defining the function ϕ by $\phi = \psi + \lambda$, we conclude that (2.9)-(2.11) of Definition 2.4 hold. To show properties (2.12)-(2.13) in the same definition, we define the measure μ_n on $\overline{\Omega} \times S(0, 1)$ as

$$\mu_n(A) = \int_{[0, T]} I_{\{(s, \phi_n(s), \gamma(s, \phi_n(s))) \in A\}} d|\lambda_n|(s),$$

for every Borel set $A \subset \overline{\Omega} \times S(0, 1)$. Introducing the notation $\overline{\Omega}_{[0, t]} := \overline{\Omega} \cap ([0, t] \times \mathbb{R}^n)$, we have, by definition and (2.13),

$$|\lambda_n|(t) = \mu_n(\overline{\Omega}_{[0, t]} \times S(0, 1)),$$

and

$$\lambda_n(t) = \int_{\overline{\Omega}_{[0, t]} \times B(0, 1)} \gamma d\mu_n(s, x, \gamma),$$

for all $t \in [0, T]$. Since $|\lambda_n|(T) \leq L < \infty$ for all n , the Banach-Alaoglu theorem asserts that a subsequence of μ_n converges to some measure μ satisfying $\mu(\overline{\Omega} \times S(0, 1)) < \infty$. By weak convergence and the continuity of λ ,

$$\lambda(t) = \int_{\overline{\Omega}_{[0, t]} \times B(0, 1)} \gamma d\mu(s, x, \gamma).$$

Define $\nu(t) = \mu(\overline{\Omega}_{[0, t]} \times S(0, 1))$. Proceeding as in the proof of Theorem 4.8 in [11], we conclude that the measure μ has the properties

$$(4.7) \quad \mu(\Sigma_1 \cup \Sigma_2) = \mu(\Sigma_3) = 0,$$

for

$$\begin{aligned} \Sigma_1 &= \Omega \times S(0, 1), \\ \Sigma_2 &= \{(s, x, \gamma) : s \in [0, T], x \in \partial\Omega_s, \gamma = \gamma(s, x)\}, \\ \Sigma_3 &= \{(s, x, \gamma) : s \in [0, T], x \neq \phi(s), \gamma \in S(0, 1)\}. \end{aligned}$$

Finally, since $\phi(0) = \psi(0)$, we have $\mu(\{0\} \times \overline{\Omega}_0 \times S(0, 1)) = 0$ and can, by the definition of ν , conclude that

$$\begin{aligned} \lambda(t) &= \int_{\{(s, x, \gamma) : s \in (0, t], x = \phi(s) \in \partial\Omega_s, \gamma = \gamma(s, \phi(s))\}} \gamma d\mu(s, x, \gamma) \\ &= \int_{(0, t]} I_{\{\phi(s) \in \partial\Omega_s\}} \widehat{\gamma}(s) d\nu(s), \end{aligned}$$

where $\widehat{\gamma}(s)$ is a ν -measurable function. Hence $d|\lambda|(s)$ is absolutely continuous with respect to $d\nu$. Together with $\mu(\Sigma_1) = 0$, this verifies (2.12). Finally, by construction, $\widehat{\gamma}(s) = \gamma(s, \phi(s))$ if $\phi(s) \in \partial\Omega_s$, $d\nu$ -a.s. Then

$$\lambda(t) = \int_{(0, t]} \widehat{\gamma}(s) d|\lambda|(t),$$

which is (2.13) in Definition 2.4. \square

5. SOLUTIONS TO REFLECTED SDES

Using the existence of solutions (ϕ, λ) to the Skorohod problem for (Ω, γ, ψ) , with $\psi \in \mathcal{C}([0, T], \mathbb{R}^n)$ and $\psi(0) \in \overline{\Omega}_0$, we are now ready to prove existence and uniqueness of solutions to SDEs with oblique reflection at the boundary of a bounded, time-dependent domain. To this end, assume that the triple (X, Y, k) satisfies the relations

$$Y(t) = x + \int_0^t b(s, X(s)) ds + \int_0^t \langle \sigma(s, X(s)), dM(s) \rangle + k(t),$$

$$X(t) \in \overline{\Omega}_t, \quad Y(t) \in \overline{\Omega}_t,$$

$$|k|(t) = \int_{(0,t]} I_{\{Y(s) \in \partial\Omega_s\}} d|k|(s) < \infty, \quad k(t) = \int_{(0,t]} \gamma(s) d|k|(s),$$

where $x \in \overline{\Omega}_0$ is fixed, $\gamma(s) = \gamma(s, Y(s)) d|k|$ -a.s. and M is a continuous \mathcal{F}_t -martingale satisfying

$$(5.1) \quad d\langle M_i, M_j \rangle(t) \leq C dt,$$

for some positive constant $C < \infty$. Let (X', Y', k') be a similar triple but with x replaced by x' .

We shall prove uniqueness of solutions by a Picard iteration scheme and a crucial ingredient is then the estimate provided by the following theorem. Note that Theorem 5.1 holds for a general continuous \mathcal{F}_t -martingale satisfying 5.1, whereas in Theorem 2.6 we restrict ourselves to letting M be a standard Wiener process.

Theorem 5.1. *There exists a positive constant $C < \infty$ such that*

$$E \left[\sup_{0 \leq s \leq t} |Y(s) - Y'(s)|^2 \right] \leq C \left(|x - x'|^2 + \int_0^t E \left[\sup_{0 \leq u \leq s} |X(u) - X'(u)|^2 \right] ds \right)$$

Proof. Fix $\varepsilon > 0$, let $\lambda > 0$ be a constant to be specified later, and let $w_\varepsilon \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^{2n}, \mathbb{R})$ and $\alpha \in \mathcal{C}^{1,2}(\overline{\Omega}, \mathbb{R})$ be the functions described in Lemma 3.2-3.3, respectively. Define the function v for all (t, x, y) such that $(t, x), (t, y) \in \overline{\Omega}$ as

$$v(t, x, y) = e^{-\lambda(\alpha(t, x) + \alpha(t, y))} w_\varepsilon(t, x, y) := u(t, x, y) w_\varepsilon(t, x, y).$$

The regularity of v is inherited from the regularity of w_ε and α . By Itô's formula we have, suppressing the s -dependence for X, X', Y and Y' ,

$$(5.2) \quad \begin{aligned} & E[v(t, Y(t), Y'(t))] \\ &= v(0, x, x') + E \int_0^t D_t v(s, Y, Y') ds \\ &+ E \int_0^t \langle D_x v(s, Y, Y'), b(s, X) \rangle ds + E \int_0^t \langle D_y v(s, Y, Y'), b(s, X') \rangle ds \\ &+ E \int_0^t \langle D_x v(s, Y, Y'), \gamma(s) \rangle d|k|(s) + E \int_0^t \langle D_y v(s, Y, Y'), \gamma(s) \rangle d|k'|(s) \\ &+ E \int_0^t \text{tr} \left(\begin{pmatrix} \sigma(s, X) \\ \sigma(s, X') \end{pmatrix}^T D^2 v(s, Y, Y') \begin{pmatrix} \sigma(s, X) \\ \sigma(s, X') \end{pmatrix} d\langle M \rangle(s) \right). \end{aligned}$$

We simplify the terms in this expression. From (3.17), (3.21) and the regularity of u we have

$$\begin{aligned} & E \int_0^t D_t v(s, Y, Y') ds \\ &= E \int_0^t D_t u(s, Y, Y') w_\varepsilon(s, Y, Y') ds + E \int_0^t u(s, Y, Y') D_t w_\varepsilon(s, Y, Y') ds \\ &\leq C(\lambda) E \int_0^t \left(\varepsilon + \frac{|Y - Y'|^2}{\varepsilon} \right) ds + C(\lambda) E \int_0^t \frac{|Y - Y'|^2}{\varepsilon} ds. \end{aligned}$$

Similarly, following the proof of Theorem 5.1 in [11], we have

$$\begin{aligned} & E \int_0^t \langle D_x v(s, Y, Y'), b(s, X) \rangle ds + E \int_0^t \langle D_y v(s, Y, Y'), b(s, X') \rangle ds \\ &\leq C(\lambda) \left(\varepsilon + E \int_0^t \frac{|Y - Y'|^2}{\varepsilon} ds + E \int_0^t \frac{|Y - Y'| |X - X'|}{\varepsilon} ds \right), \end{aligned}$$

and

$$\begin{aligned}
& E \int_0^t \langle D_x v(s, Y, Y'), \gamma(s) \rangle d|k|(s) + E \int_0^t \langle D_y v(s, Y, Y'), \gamma(s) \rangle d|k'|(s) \\
& \leq CE \int_0^t u(s, Y, Y') \frac{|Y - Y'|^2}{\varepsilon} d|k|(s) + CE \int_0^t u(s, Y, Y') \frac{|Y - Y'|^2}{\varepsilon} d|k'|(s) \\
& \quad - \lambda E \int_0^t v(s, Y, Y') \langle D_x \alpha(s, Y), \gamma(s) \rangle d|k|(s) \\
& \quad - \lambda E \int_0^t v(s, Y, Y') \langle D_x \alpha(s, Y'), \gamma(s) \rangle d|k'|(s).
\end{aligned}$$

A simple extension of Lemma 5.7 in [11] to the time-dependent case shows that there exists a constant $K_1(\lambda) < \infty$ such that for all $t \in [0, T]$, $x, y \in \bar{\Omega}_t$, the second order derivatives of v with respect to the spatial variables satisfy

$$D^2 v(t, x, y) \leq K_1(\lambda) \left(\frac{1}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \left(\varepsilon + \frac{|x - y|^2}{\varepsilon} \right) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right).$$

Moreover, it is an easy consequence of the Lipschitz continuity of σ that there exists a constant $K_2(\lambda) < \infty$ such that for all $t \in [0, T]$, $x, y, \xi, \omega \in \bar{\Omega}_t$,

$$\begin{pmatrix} \sigma(t, \xi) \\ \sigma(t, \omega) \end{pmatrix}^T D^2 v(t, x, y) \begin{pmatrix} \sigma(t, \xi) \\ \sigma(t, \omega) \end{pmatrix} \leq K_2(\lambda) \left(\varepsilon + \frac{1}{\varepsilon} (|\xi - \omega|^2 + |x - y|^2) \right) I.$$

Consequently, the final term in (5.2) may be simplified to

$$\begin{aligned}
& E \int_0^t \text{tr} \left(\begin{pmatrix} \sigma(s, X) \\ \sigma(s, X') \end{pmatrix}^T D^2 v(s, Y, Y') \begin{pmatrix} \sigma(s, X) \\ \sigma(s, X') \end{pmatrix} \right) d\langle M \rangle(s) \\
& \leq C(\lambda) \left(\varepsilon + E \int_0^t \frac{|X - X'|^2}{\varepsilon} ds + E \int_0^t \frac{|Y - Y'|^2}{\varepsilon} ds \right).
\end{aligned}$$

Returning to the terms containing $|k|$ and $|k'|$, we note that, by (3.16) and (3.24), we have

$$\begin{aligned}
& -\lambda E \int_0^t v(s, Y, Y') \langle D_x \alpha(s, Y), \gamma(s) \rangle d|k|(s) \\
& \leq -\lambda E \int_0^t u(s, Y, Y') w_\varepsilon(s, Y, Y') d|k|(s) \\
& \leq -\lambda \chi E \int_0^t u(s, Y, Y') \frac{|Y - Y'|^2}{\varepsilon} d|k|(s),
\end{aligned}$$

so, by putting $\lambda = C/\chi$ all integrals with respect to $|k|$ and $|k'|$ vanish. Dropping the λ -dependence from the constants, (5.2) can be simplified to

$$\begin{aligned}
& E \left[\frac{|Y(t) - Y'(t)|^2}{\varepsilon} \right] \\
& \leq CE[v(t, Y(t), Y'(t))] \leq Cv(0, x, x') + C\varepsilon + CE \int_0^t \frac{|Y - Y'|^2}{\varepsilon} ds \\
& \quad + CE \int_0^t \frac{|X - X'| |Y - Y'|}{\varepsilon} ds + CE \int_0^t \frac{|X - X'|^2}{\varepsilon} ds \\
& \leq C \frac{|x - x'|^2}{\varepsilon} + C\varepsilon + CE \int_0^t \left(\frac{|X - X'|^2}{\varepsilon} + \frac{|Y - Y'|^2}{\varepsilon} \right) ds,
\end{aligned}$$

which, after multiplying with ε and letting ε tend to zero, gives

$$E \left[|Y(t) - Y'(t)|^2 \right] \leq C \left(|x - x'|^2 + \int_0^t E \left[|X - X'|^2 + |Y - Y'|^2 \right] ds \right).$$

The requested inequality now follows by a simple application of Doob's and Gronwall's inequalities (see Theorem 5.1 in [11] for details). \square

Proof of Theorem 2.6. Given Theorems 4.5 and 5.1, the proof of Theorem 2.6 follows exactly along the lines of the proof of Corollary 5.2 in [11]. We omit further details. \square

6. SOLUTIONS TO FULLY NONLINEAR SECOND-ORDER PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

In this section, we prove the results on partial differential equations. First, we recall the definition of viscosity solutions. Let $E \subset \mathbb{R}^{n+1}$ be arbitrary. If $u : E \rightarrow \mathbb{R}$, then the parabolic superjet $\mathcal{P}_E^{2,+}u(s, z)$ contains all triplets $(a, p, X) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ such that if $(s, z) \in E$ then

$$\begin{aligned} u(t, x) &\leq u(s, z) + a(t - s) + \langle p, x - z \rangle + \frac{1}{2} \langle X(x - z), x - z \rangle \\ &\quad + o(|t - s| + |x - z|^2) \quad \text{as } E \ni (t, x) \rightarrow (s, z). \end{aligned}$$

The parabolic subjet is defined as $\mathcal{P}_E^{2,-}u(s, z) = -\mathcal{P}_E^{2,+}(-u(s, z))$. The closures $\overline{\mathcal{P}}_E^{2,+}u(s, z)$ and $\overline{\mathcal{P}}_E^{2,-}u(s, z)$ are defined in analogue with (2.6) and (2.7) in [8]. Moreover, let $USC(E)$ ($LSC(E)$) denote the set of upper (lower) semi-continuous functions on E . A function $u \in USC(\tilde{\Omega})$ is a *viscosity subsolution* of (2.19) in Ω° if, for all $(a, p, A) \in \mathcal{P}_\Omega^{2,+}u(t, x)$, it holds that

$$a + F(t, x, u(t, x), p, A) \leq 0 \quad \text{for } (t, x) \in \Omega^\circ,$$

If, in addition, for $(t, x) \in \partial\Omega$ it holds that

$$\min\{a + F(t, x, u(t, x), p, A), \langle \tilde{\gamma}(t, x), p \rangle + f(t, x, u(t, x))\} \leq 0.$$

then u is a viscosity subsolution of (2.19)-(2.20) in $\tilde{\Omega}$. Similarly, a function $v \in LSC(\tilde{\Omega})$ is a *viscosity supersolution* of (2.19) in Ω° if, for all $(a, p, A) \in \mathcal{P}_\Omega^{2,-}v(t, x)$, it holds that

$$a + F(t, x, v(t, x), p, A) \geq 0 \quad \text{for } (t, x) \in \Omega^\circ.$$

If, in addition, for $(t, x) \in \partial\Omega$ it holds that

$$\max\{a + F(t, x, v(t, x), p, A), \langle \tilde{\gamma}(t, x), p \rangle + f(t, x, v(t, x))\} \geq 0,$$

then v is a viscosity supersolution of (2.19)-(2.20) in $\tilde{\Omega}$. A function is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution. We remark that in the definition of viscosity solutions above, we may replace $\mathcal{P}_\Omega^{2,+}u(t, x)$ and $\mathcal{P}_\Omega^{2,-}v(t, x)$ by $\overline{\mathcal{P}}_\Omega^{2,+}u(t, x)$ and $\overline{\mathcal{P}}_\Omega^{2,-}v(t, x)$, respectively. In the following, we often skip writing "viscosity" before subsolutions, supersolutions and solutions.

In the proofs below we will make use of the matrix norm notation

$$\|A\| = \sup\{|\lambda| : \lambda \text{ is an eigenvalue of } A\} = \sup\{|\langle A\xi, \xi \rangle| : |\xi| \leq 1\},$$

and for symmetric matrices $X, Y \in \mathbb{S}^n$, we write $X \leq Y$ if $\langle (X - Y)\xi, \xi \rangle \leq 0$ for all $\xi \in \mathbb{R}^n$. Note also that, given any set $E \subset \mathbb{R}^{n+1}$ and $t \in [0, T]$, we denote, in the following, the time sections of E as $E_t = \{x : (t, x) \in E\}$.

Next we give two lemmas. The first clarifies that the maximum principle for semicontinuous functions [7], [8], holds true in time-dependent domains.

Lemma 6.1. *Suppose that $\mathcal{O}^i = \widehat{\mathcal{O}}^i \cap ((0, T) \times \mathbb{R}^n)$ for $i = 1, \dots, k$ where $\widehat{\mathcal{O}}^i$ are locally compact subsets of \mathbb{R}^{n+1} . Assume that $u_i \in USC(\mathcal{O}^i)$ and let $\varphi : (t, x_1, \dots, x_k) \rightarrow \varphi(t, x_1, \dots, x_k)$ be defined on an open neighborhood of $\{(t, x) : t \in (0, T) \text{ and } x_i \in \mathcal{O}_t^i \text{ for } i = 1, \dots, k\}$ and such that φ is once continuously differentiable in t and twice continuously differentiable in (x_1, \dots, x_k) . Suppose that $s \in (0, T)$ and $z_i \in \mathcal{O}_s^i$ and*

$$w(t, x_1, \dots, x_k) \equiv u_1(t, x_1) + \dots + u_k(t, x_k) - \varphi(t, x_1, \dots, x_k) \leq w(s, z_1, \dots, z_k)$$

for $0 < t < T$ and $x_i \in \mathcal{O}_t^i$. Assume, moreover, that there is an $r > 0$ such that for every $M > 0$ there is a C such that, for $i = 1, \dots, k$,

$$(6.1) \quad b_i \leq C, \text{ whenever } (b_i, q_i, X_i) \in \mathcal{P}_{\mathcal{O}_t^i}^{2,+} u_i(t, x) \text{ with } \|X_i\| \leq M \text{ and} \\ |x_i - z_i| + |t - s| + |u_i(t, x_i) - u_i(s, z_i)| + |q_i - D_{x_i} \varphi(s, z_1, \dots, z_k)| \leq r.$$

Then, for each $\varepsilon > 0$ there exist (b_i, X_i) such that

$$(b_i, D_{x_i} \varphi(s, z_1, \dots, z_k), X_i) \in \overline{\mathcal{P}}_{\mathcal{O}^i}^{2,+} u_i(s, z) \text{ for } i = 1, \dots, k, \\ - \left(\frac{1}{\varepsilon} + \|A\| \right) I \leq \begin{pmatrix} X_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & X_k \end{pmatrix} \leq A + \varepsilon A^2$$

and

$$b_1 + \dots + b_k = D_t \varphi(s, z_1, \dots, z_k),$$

where $A = (D_x^2 \varphi)(s, z_1, \dots, z_k)$.

Proof. Following ideas from page 1008 in [7] we let K_i be compact neighborhoods of (s, z) in \mathcal{O}^i and define the extended functions $\tilde{u}_1, \dots, \tilde{u}_k, \tilde{u}_i \in USC(\mathbb{R}^n)$ for $i = 1, \dots, k$, by

$$\tilde{u}_i(t, x) = \begin{cases} u_i(t, x) & \text{if } (t, x) \in K_i \\ -\infty & \text{otherwise.} \end{cases}$$

From the definitions of sub and superjets it follows, for $i = 1, \dots, k$, that

$$(6.2) \quad \mathcal{P}_{\mathbb{R}^{n+1}}^{2,+} \tilde{u}_i(t, x) = \mathcal{P}_{\mathcal{O}^i}^{2,+} u_i(t, x)$$

for (t, x) in the interior of K_i relative to \mathcal{O}^i . Excluding the trivial case $u_i(t, x) = -\infty$, then the function $\tilde{u}_i(t, x)$ cannot approach $u_i(s, z)$ unless $(t, x) \in K_i$ and it follows that

$$(6.3) \quad \overline{\mathcal{P}}_{\mathbb{R}^{n+1}}^{2,+} \tilde{u}_i(t, x) = \overline{\mathcal{P}}_{\mathcal{O}^i}^{2,+} u_i(t, x).$$

Setting $\tilde{w}(t, x_1, \dots, x_k) = \tilde{u}_1(t, x_1) + \dots + \tilde{u}_k(t, x_k)$ we see that (s, z_1, \dots, z_k) is also a maximum of the function $(\tilde{w} - \varphi)(t, x_1, \dots, x_k)$. Moreover, we note that the proof of Lemma 8 in [7] still works if (27) in [7] is replaced by assumption (6.1). These facts, together with (6.2) and (6.3), allows us to complete the proof of Lemma 6.1 by using Theorem 7 in [7]. \square

Before proving the next lemma, let us note that standard arguments imply that we can assume $\lambda > 0$ in (2.22). Indeed, if $\lambda \leq 0$ then for $\bar{\lambda} < \lambda$ the functions $e^{\bar{\lambda}t} u(t, x)$ and $e^{\bar{\lambda}t} v(t, x)$ are, respectively, sub and supersolutions of (2.19) and (2.20) with $F(t, x, r, p, X)$ and $f(t, x, r)$ replaced by

$$(6.4) \quad -\bar{\lambda}r + e^{\bar{\lambda}t} F(t, x, e^{-\bar{\lambda}t} r, e^{-\bar{\lambda}t} p, e^{-\bar{\lambda}t} X) \quad \text{and} \quad e^{\bar{\lambda}t} f(t, x, e^{-\bar{\lambda}t} r).$$

Hence, in the following proof we assume $\lambda > 0$ in (2.22). Next we prove the following version of the maximum principle.

Lemma 6.2. *Let Ω° be a time-dependent domain as in (2.18). Assume (2.21)-(2.23). Let u and v be, respectively, a viscosity subsolution and a viscosity supersolution of (2.19) in Ω° . Then $\sup_{\bar{\Omega}} u - v = \sup_{\partial\Omega \cup \bar{\Omega}_0} (u - v)^+$.*

Proof. We may assume, by replacing $T > 0$ by a smaller number if necessary, that u and $-v$ are bounded from above on $\tilde{\Omega}$. We can also assume that $\sup_{\tilde{\Omega}} u - v$ is attained by using the well known fact that if u is a subsolution of (2.19), then so is

$$u_\beta(t, x) = u(t, x) - \frac{\beta}{T - t},$$

for all $\beta > 0$. Hence, it suffices to prove the lemma for all $\beta > 0$. Assume that $\sup_{\tilde{\Omega}} u - v = u(s, z) - v(s, z) > u(t, x) - v(t, x)$ for some $(s, z) \in \Omega^\circ$ and for all $(t, x) \in \partial\Omega \cup \bar{\Omega}_0$. As in Section 5.B in [8], we use the fact that if u is a viscosity subsolution, then so is $\bar{u} = u - K$ for every constant $K > 0$. Choose $K > 0$ such that $\bar{u}(t, x) - v(t, x) \leq 0$ for all $(t, x) \in \partial\Omega \cup \bar{\Omega}_0$ and such that $\bar{u}(s, z) - v(s, z) := \delta > 0$. Using Lemma 6.1 in place of Theorem 8.3 in [8] and by observing that assumptions (2.21)-(2.23) imply (assuming $\lambda > 0$ as is possible by (6.4)) the corresponding assumptions in [8], we see that we can proceed as in the proof of Theorem 8.2 in [8] to complete the proof. \square

6.1. Proof of Theorem 2.7. We begin by proving the comparison principle in Theorem 2.7. In the following we may assume, by replacing $T > 0$ by a smaller number if necessary, that u and $-v$ in Theorem 2.7 are bounded from above on $\tilde{\Omega}$. We will now produce approximations of u and v which allows us to deal only with the inequalities involving F and not the boundary conditions. To construct these approximating functions, we note that Lemma 3.3 applies with γ replaced by $\tilde{\gamma}$ as well. Thus, there exists a $C^{1,2}$ function α defined on an open neighborhood of $\tilde{\Omega}$ with the property that $\alpha \geq 0$ on $\tilde{\Omega}$ and $\langle \tilde{\gamma}(t, x), D_x \alpha(t, x) \rangle \geq 1$ for $x \in \partial\Omega_t$, $t \in (0, T)$. For $\beta_1 > 0$, $\beta_2 > 0$ and $\beta_3 > 0$ we define, for $(t, x) \in \tilde{\Omega} \cup \bar{\Omega}_0$,

$$(6.5) \quad \begin{aligned} u_{\beta_1, \beta_2, \beta_3}(t, x) &= u(t, x) - \beta_1 \alpha(t, x) - \beta_2 - \frac{\beta_3}{T - t}, \\ v_{\beta_1, \beta_2}(t, x) &= v(t, x) + \beta_1 \alpha(t, x) + \beta_2. \end{aligned}$$

Given $\beta_3, \beta_2 > 0$ there is $\beta_1 = \beta_1(\beta_2) \in (0, \beta_2)$ for which $u_{\beta_1, \beta_2, \beta_3}$ and v_{β_1, β_2} are sub- and supersolutions of (2.19) and (2.20), with $f(t, x, r)$ replaced by $f(t, x, r) + \beta_1$ and $f(t, x, r) - \beta_1$, respectively. Indeed, if $(a, p, X) \in \mathcal{P}_{\tilde{\Omega}}^{2,+} u_{\beta_1, \beta_2, \beta_3}(t, x)$, then

$$(6.6) \quad \left(a + \beta_1 \alpha_t(t, x) + \frac{\beta_3}{(T - t)^2}, p + \beta_1 D\alpha(t, x), X + \beta_1 D^2 \alpha(t, x) \right) \in \mathcal{P}_{\tilde{\Omega}}^{2,+} u(t, x).$$

Hence, if u satisfies (2.20), then $\langle \tilde{\gamma}(t, x), p + \beta_1 D\alpha(t, x) \rangle + f(t, x, u(t, x)) \leq 0$ and since $\langle \tilde{\gamma}(t, x), D\alpha(t, x) \rangle \geq 1$, $u_{\beta_1, \beta_2, \beta_3} \leq u$ and by (2.26) we obtain

$$(6.7) \quad \langle \tilde{\gamma}(t, x), p \rangle + f(t, x, u_{\beta_1, \beta_2, \beta_3}) + \beta_1 \leq 0.$$

Using (6.6) we also see that if u satisfies (2.19) then

$$a + \beta_1 \alpha_t(t, x) + \frac{\beta_3}{(T - t)^2} + F(t, x, u, p + \beta_1 D\alpha(t, x), X + \beta_1 D^2 \alpha(t, x)) \leq 0.$$

Using (2.22) and (2.24), assuming also that the support of α lies within U , we have

$$(6.8) \quad \begin{aligned} a + \beta_1 \alpha_t(t, x) + F(t, x, u_{\beta_1, \beta_2, \beta_3}, p, X) + \lambda \beta_2 \\ - m_2 (|\beta_1 D\alpha(t, x)| + |\beta_1 D^2 \alpha(t, x)|) \leq 0. \end{aligned}$$

From (6.7) and (6.8) it follows that, given $\beta_2, \beta_3 > 0$, there exist $\beta_1 \in (0, \beta_2)$ such that $u_{\beta_1, \beta_2, \beta_3}$ is a subsolution of (2.19) and (2.20) with $f(t, x, u)$ replaced by $f(t, x, u) + \beta_1$. The fact that v_{β_1, β_2} is a supersolution follows by a similar calculation.

To complete the proof of the comparison principle, it is sufficient to prove that

$$\max_{\tilde{\Omega}} (u_{\beta_1, \beta_2, \beta_3} - v_{\beta_1, \beta_2}) \leq 0$$

holds for all $\beta_2 > 0$ and $\beta_3 > 0$. Assume that

$$\sigma = \max_{\tilde{\Omega}} (u_{\beta_1, \beta_2, \beta_3} - v_{\beta_1, \beta_2}) > 0.$$

We will derive a contradiction for any β_3 if β_2 (and hence β_1) is small enough. To simplify notation, we write, in the following, u, v in place of $u_{\beta_1, \beta_2, \beta_3}, v_{\beta_1, \beta_2}$. By Lemma 6.2, upper semicontinuity of $u - v$ and boundedness from above of $u - v$, we conclude that for any $\beta_3 > 0$

$$\sigma = (u - v)(s, z) \quad \text{for some } z \in \partial\Omega_s \text{ and } s \in (0, T).$$

Let $\tilde{B}((s, z), \delta) = \{(t, x) : |(t, x) - (s, z)| \leq \delta\}$ and define

$$E := \tilde{B}((s, z), \delta) \cap \tilde{\Omega}.$$

By Remark 2.1, there exists $\theta \in (0, 1)$ such that

$$(6.9) \quad \langle x - y, \tilde{\gamma}(t, x) \rangle \geq -\theta |x - y| \quad \text{for all } (t, x) \in E \setminus \Omega^\circ \text{ and } (t, y) \in E.$$

By decreasing δ if necessary, we may assume that (2.24) holds in E . From now on, we restrict our attention to events in the set E . By Lemma 3.2 we obtain, for any $\theta \in (0, 1)$, a family $\{w_\varepsilon\}_{\varepsilon > 0}$ of functions $w_\varepsilon \in C^{1,2}([0, T] \times \mathbb{R}^{2n}, \mathbb{R})$ and positive constants χ, C (independent of ε) such that

$$(6.10) \quad w_\varepsilon(t, x, y) \geq \chi \frac{|x - y|^2}{\varepsilon},$$

$$(6.11) \quad w_\varepsilon(t, x, y) \leq C \left(\varepsilon + \frac{|x - y|^2}{\varepsilon} \right),$$

$$(6.12) \quad \langle D_x w_\varepsilon(t, x, y), \tilde{\gamma}(t, x) \rangle \geq -C \frac{|x - y|^2}{\varepsilon}, \quad \text{if } \langle x - y, \tilde{\gamma}(t, x) \rangle \geq -\theta |x - y|,$$

$$(6.13) \quad \langle D_y w_\varepsilon(t, x, y), \tilde{\gamma}(t, y) \rangle \geq 0, \quad \text{if } \langle y - x, \tilde{\gamma}(t, y) \rangle \geq -\theta |x - y|,$$

$$(6.14) \quad |D_t w_\varepsilon(t, x, y)| \leq C \frac{|x - y|^2}{\varepsilon},$$

$$(6.15) \quad |D_y w_\varepsilon(t, x, y)| \leq C \frac{|x - y|}{\varepsilon}, \quad |D_x w_\varepsilon(t, x, y) + D_y w_\varepsilon(t, x, y)| \leq C \frac{|x - y|^2}{\varepsilon},$$

$$(6.16) \quad D^2 w_\varepsilon(t, x, y) \leq \frac{C}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \frac{C|x - y|^2}{\varepsilon} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

for $t \in [0, T]$, x, y belong to a compactly supported subset of \mathbb{R}^n .

Let $\varepsilon > 0$ be given and define

$$\Phi(t, x, y) = u(t, x) - v(t, y) - \varphi(t, x, y),$$

where

$$\varphi(t, x, y) = w_\varepsilon(t, x, y) + f(s, z, u(s, z)) \langle \tilde{\gamma}(s, z), y - x \rangle + \beta_1 |x - z|^2 + (t - s)^2.$$

Let $(t_\varepsilon, x_\varepsilon, y_\varepsilon)$ be a maximum point of Φ . From (6.10) and (6.11) we have

$$(6.17) \quad \begin{aligned} \sigma - C\varepsilon \leq \Phi(s, z, z) &\leq \Phi(t_\varepsilon, x_\varepsilon, y_\varepsilon) \leq u(t_\varepsilon, x_\varepsilon) - v(t_\varepsilon, y_\varepsilon) - \chi \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \\ &\quad - f(s, z, u(s, z)) \langle \tilde{\gamma}(s, z), y_\varepsilon - x_\varepsilon \rangle - \beta_1 |x_\varepsilon - z|^2 - (t_\varepsilon - s)^2. \end{aligned}$$

From this we first see that

$$|x_\varepsilon - y_\varepsilon| \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.$$

Therefore, using the upper semi-continuity of $u - v$ and (6.17) we also obtain

$$(6.18) \quad \begin{aligned} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} &\rightarrow 0, & x_\varepsilon, y_\varepsilon &\rightarrow z, & t_\varepsilon &\rightarrow s, \\ u(t_\varepsilon, x_\varepsilon) &\rightarrow u(s, z), & v(t_\varepsilon, y_\varepsilon) &\rightarrow v(s, z) \end{aligned}$$

as $\varepsilon \rightarrow 0$. In the following we assume ε to be so small that $(t_\varepsilon, x_\varepsilon) \in E$

We introduce the notation

$$\begin{aligned} \bar{p} &= D_x \varphi(t_\varepsilon, x_\varepsilon, y_\varepsilon) = D_x w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) - f(s, z, u(s, z)) \tilde{\gamma}(s, z) + 2\beta_1 (x_\varepsilon - z), \\ \bar{q} &= D_y \varphi(t_\varepsilon, x_\varepsilon, y_\varepsilon) = D_y w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) + f(s, z, u(s, z)) \tilde{\gamma}(s, z) \end{aligned}$$

and observe that

$$(6.19) \quad \begin{aligned} &\langle \tilde{\gamma}(t_\varepsilon, x_\varepsilon), \bar{p} \rangle + f(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon)) \\ &= \langle \tilde{\gamma}(t_\varepsilon, x_\varepsilon), D_x w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) \rangle + f(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon)) \\ &\quad - f(s, z, u(s, z)) \langle \tilde{\gamma}(t_\varepsilon, x_\varepsilon), \tilde{\gamma}(s, z) \rangle + 2\beta_1 \langle \tilde{\gamma}(t_\varepsilon, x_\varepsilon), x_\varepsilon - z \rangle, \end{aligned}$$

and

$$(6.20) \quad \begin{aligned} &-\langle \tilde{\gamma}(t_\varepsilon, y_\varepsilon), \bar{q} \rangle + f(t_\varepsilon, y_\varepsilon, v(t_\varepsilon, y_\varepsilon)) \\ &= -\langle \tilde{\gamma}(t_\varepsilon, y_\varepsilon), D_y w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) \rangle + f(t_\varepsilon, y_\varepsilon, v(t_\varepsilon, y_\varepsilon)) \\ &\quad - f(s, z, u(s, z)) \langle \tilde{\gamma}(t_\varepsilon, y_\varepsilon), \tilde{\gamma}(s, z) \rangle. \end{aligned}$$

Using (2.3), (2.25), (2.26) and (6.18)-(6.20) we see that if ε is small enough, then

$$(6.21) \quad \begin{aligned} &\langle \tilde{\gamma}(t_\varepsilon, x_\varepsilon), D_x w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) \rangle \geq -\frac{\beta_1}{2} \\ &\implies \langle \tilde{\gamma}(t_\varepsilon, x_\varepsilon), \bar{p} \rangle + f(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon)) + \beta_1 > 0, \\ &\langle \tilde{\gamma}(t_\varepsilon, y_\varepsilon), D_y w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) \rangle \geq -\frac{\beta_1}{2} \\ &\implies -\langle \tilde{\gamma}(t_\varepsilon, y_\varepsilon), \bar{q} \rangle + f(t_\varepsilon, y_\varepsilon, v(t_\varepsilon, y_\varepsilon)) - \beta_1 < 0. \end{aligned}$$

Moreover, from (6.9) and (6.12)-(6.13), we also have

$$(6.22) \quad \begin{aligned} &\langle \tilde{\gamma}(t_\varepsilon, x_\varepsilon), D_x w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) \rangle \geq -C \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon}, & \text{if } x_\varepsilon \in \partial\Omega_{t_\varepsilon}, \\ &\langle \tilde{\gamma}(t_\varepsilon, y_\varepsilon), D_y w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) \rangle \geq 0, & \text{if } y_\varepsilon \in \partial\Omega_{t_\varepsilon}. \end{aligned}$$

Using (6.21)-(6.22), it follows by the definition of viscosity solutions that if ε is small enough, say $0 < \varepsilon < \varepsilon_{\beta_1}$, then

$$(6.23) \quad a + F(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon), \bar{p}, X) \leq 0 \leq -b + F(t_\varepsilon, y_\varepsilon, v(t_\varepsilon, y_\varepsilon), -\bar{q}, -Y)$$

whenever

$$(a, \bar{p}, X) \in \overline{\mathcal{P}}_\Omega^{2,+} u(t_\varepsilon, x_\varepsilon) \quad \text{and} \quad (-b, -\bar{q}, -Y) \in \overline{\mathcal{P}}_\Omega^{2,-} v(t_\varepsilon, y_\varepsilon).$$

We next intend to use Lemma 6.1 to show the existence of such matrices X, Y and numbers a, b . Hence, we have to verify condition (6.1). To do so, we observe that (6.21) holds true with \bar{p} and \bar{q} replaced by any p and q satisfying $|\bar{p} - p| \leq r$ and $|\bar{q} - q| \leq r$ if we chose $r = r(\varepsilon)$ small enough. It follows that also (6.23) holds with these p and q and we can conclude

$$a \leq -F(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon), p, X) \leq C \quad \text{and} \quad b \leq F(t_\varepsilon, y_\varepsilon, v(t_\varepsilon, y_\varepsilon), -q, -Y) \leq C$$

for some $C = C(\varepsilon)$ whenever (a, p, X) and (b, q, Y) is as in (6.1). Hence, condition (6.1) holds and Lemma 6.1 gives the existence of $X, Y \in \mathbb{S}^n$ and $a, b \in \mathbb{R}$ such that

$$(6.24) \quad \begin{aligned} -\left(\frac{1}{\varepsilon} + \|A\|\right) I &\leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq A + \varepsilon A^2, \\ (a, \bar{p}, X) &\in \overline{\mathcal{P}}_{\Omega}^{2,+} u(t_\varepsilon, x_\varepsilon), \quad (-b, -\bar{q}, -Y) \in \overline{\mathcal{P}}_{\Omega}^{2,-} v(t_\varepsilon, y_\varepsilon), \\ a + b &= D_t \varphi(t_\varepsilon, x_\varepsilon, y_\varepsilon) = D_t w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) + 2(t_\varepsilon - s), \end{aligned}$$

where $A = D_{x,y}^2(w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) + \beta_1|x_\varepsilon - z|^2)$. Using (2.22), (6.14) and (6.23) we obtain, by recalling that we can assume $\lambda > 0$ in (2.22), that

$$\begin{aligned} 0 &\geq D_t w_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) + 2(t_\varepsilon - s) \\ &\quad + F(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon), \bar{p}, X) - F(t_\varepsilon, y_\varepsilon, v(t_\varepsilon, y_\varepsilon), -\bar{q}, -Y) \\ &\geq -C \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} + 2(t_\varepsilon - s) + \lambda(u(t_\varepsilon, x_\varepsilon) - v(t_\varepsilon, y_\varepsilon)) \\ &\quad + F(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon), \bar{p}, X) - F(t_\varepsilon, y_\varepsilon, u(t_\varepsilon, x_\varepsilon), -\bar{q}, -Y). \end{aligned}$$

Next, assumption (2.24) gives

$$(6.25) \quad \begin{aligned} 0 &\geq -C\bar{s} + 2(t_\varepsilon - s) + \lambda(u(t_\varepsilon, x_\varepsilon) - v(t_\varepsilon, y_\varepsilon)) \\ &\quad + F(t_\varepsilon, x_\varepsilon, u(t_\varepsilon, x_\varepsilon), -\bar{q}, X - C\bar{s}I) - F(t_\varepsilon, y_\varepsilon, u(t_\varepsilon, x_\varepsilon), -\bar{q}, -Y + C\bar{s}I) \\ &\quad - m_2(|\bar{p} + \bar{q}| + C\bar{s}) - m_2(C\bar{s}) \end{aligned}$$

where we use the notation $\bar{s} = |x_\varepsilon - y_\varepsilon|^2/\varepsilon$. Note that since the eigenvalues of εA^2 are given by $\varepsilon \lambda^2$, where λ is an eigenvalue to A , and since λ is bounded, $A + \varepsilon A^2 \leq CA$. Hence, by (6.16) we obtain

$$A + \varepsilon A^2 \leq \frac{C}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C\bar{s}I_{2n},$$

and since $\|A\| \leq C/\varepsilon$ for some large C , we also conclude that (6.24) implies

$$-\frac{C}{\varepsilon}I_{2n} \leq \begin{pmatrix} X - C\bar{s} & 0 \\ 0 & Y - C\bar{s} \end{pmatrix} \leq \frac{C}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Using the above inequality, assumption (2.23), (6.25), the definition of \bar{q} and (6.15) we have

$$\begin{aligned} 0 &\geq -C\bar{s} + 2(t_\varepsilon - s) + \lambda(u(t_\varepsilon, x_\varepsilon) - v(t_\varepsilon, y_\varepsilon)) \\ &\quad - m_1(C|x_\varepsilon - y_\varepsilon| + 2C\bar{s}) - m_2(|\bar{p} + \bar{q}| + C\bar{s}) - m_2(C\bar{s}) \end{aligned}$$

when $0 < \varepsilon < \varepsilon_{\beta_1}$ and $u(t_\varepsilon, x_\varepsilon) \geq v(t_\varepsilon, y_\varepsilon)$. Sending first ε and then β_2 to zero (the latter implies $\beta_1 \rightarrow 0$) and using (6.15) we obtain a contradiction. This completes the proof of the comparison principle in Theorem 2.7.

To prove existence we will use Perron's method. To do so we let u^* and u_* denote the upper and lower semicontinuous envelope of a function u , respectively. Let u and v be as in the statement of Theorem 2.7 and define

$$\tilde{w} := \sup\{w(x) : w \text{ is a subsolution of (2.19), (2.20) and (2.28)}\}.$$

By the comparison principle and by construction we obtain

$$u_* \leq \tilde{w}_* \leq \tilde{w}^* \leq v^*$$

on $\tilde{\Omega} \cup \overline{\Omega}_0$. Since $u(0, x) = v(0, x) = g(x)$ for all $x \in \overline{\Omega}_0$ and g is continuous, we also conclude that \tilde{w}^* satisfies the initial conditions of being a subsolution and \tilde{w}_* satisfies the initial conditions of being a supersolution, that is

$$(6.26) \quad \tilde{w}^*(0, x) \leq g(x) \leq \tilde{w}_*(0, x) \quad \text{for all } x \in \overline{\Omega}_0.$$

We can now proceed as in [8] (see also [2] and [14]) to show that \tilde{w}^* is a viscosity subsolution and \tilde{w}_* is a viscosity supersolution of (2.19), (2.20) and (2.28). Using

the comparison principle once again, we then have $\tilde{w}_* \geq \tilde{w}^*$ and so $\tilde{w}_* = \tilde{w}^*$ is our sought viscosity solution. This completes the proof of Theorem 2.7. \square

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